

JOURNAL OF FUNCTIONAL ANALYSIS **59**, 572–611 (1984)

Layer Potentials and Regularity for the Dirichlet Problem for Laplace's Equation in Lipschitz Domains

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Communicated by Peter D. Lax

Received March 1, 1983; revised April 16, 1984

For D , a bounded Lipschitz domain in \mathbf{R}^n , $n \geq 2$, the classical layer potentials for Laplace's equation are shown to be invertible operators on $L^2(\partial D)$ and various subspaces of $L^2(\partial D)$. For $1 < p \leq 2$ and data in $L^p(\partial D)$ with first derivatives in $L^p(\partial D)$ it is shown that there exists a unique harmonic function, u , that solves the Dirichlet problem for the given data and such that the nontangential maximal function of ∇u is in $L^p(\partial D)$. When $n = 2$ the question of the invertibility of the layer potentials on every $L^p(\partial D)$, $1 < p < \infty$, is answered. © 1984 Academic Press, Inc.

Contents. Introduction. 0. Some definitions and known results. 1. Trace theorems for potentials. 2. An operator inequality. 3. Invertibility of layer potentials in L^2 . 4. Invertibility of layer potentials in L^p for Lipschitz domains in the plane. 5. Regularity for the Dirichlet problem.

INTRODUCTION

This paper is concerned with the invertibility of classical layer potentials for Laplace's equation on the boundaries of bounded Lipschitz domains in \mathbf{R}^n and the application of these potentials to the Dirichlet and Neumann problems. For simplicity any bounded Lipschitz domain, D , considered in this paper will be assumed to have connected boundary. Thus both D and $\mathbf{R}^n \setminus \bar{D}$ will be connected open sets. The spaces of boundary functions with which we will be concerned are the Lebesgue spaces, $L^p(\partial D)$, with respect to surface measure, σ , and the spaces, $L_1^p(\partial D)$, of $L^p(\partial D)$ functions with first derivatives in $L^p(\partial D)$.

The main results to be found in Sections 3 and 5 are the invertibility of various potentials on $L^2(\partial D)$ and subspaces of $L^2(\partial D)$ and a result we will call regularity for the Dirichlet problem. The latter may be stated as follows. If $1 < p \leq 2$ and data are taken in $L_1^p(\partial D)$ then there exists a unique harmonic function defined in $D \subset \mathbf{R}^n$ that has the given data for its Dirichlet

data via a nontangential approach to ∂D and such that the nontangential maximal function of the gradient of this harmonic function is in $L^p(\partial D)$.

In the 1978 paper of E. B. Fabes, M. Jodeit, Jr., and N. M. Rivi  re [9], it was shown, in the case of C^1 domains, using layer potential techniques, that there existed unique solutions to the Dirichlet and Neumann problems posed with boundary data in L^p whenever $1 < p < \infty$. Regularity for the Dirichlet problem was demonstrated for $1 < p < \infty$. An important ingredient of their proofs was A. P. Calder  n's result [2] on the boundedness of the Cauchy integral on curves in \mathbf{R}^2 with small Lipschitz norm. The restriction on the size of the Lipschitz norm was one impediment to the application of potential techniques to general Lipschitz domains. However, B. E. J. Dahlberg [6, 7], following the work of Hunt and Wheeden [11, 12], was able to show solvability of the Dirichlet problem for $2 \leq p \leq \infty$ by examining the Poisson kernel for these domains. That this was the best possible range of L^p spaces was a well-known fact. See [10]. Recently D. S. Jerison and C. E. Kenig [13] showed existence and uniqueness up to constants for solutions to the Neumann problem with data in L^2 and showed regularity for the Dirichlet problem with data in L^2_1 .

In the spring of 1981, R. R. Coifman, A. McIntosh, and Y. Meyer [5] showed that the Cauchy integral was indeed a bounded operator for L^p , $1 < p < \infty$, on curves with arbitrarily large Lipschitz norm. In Section 1 we state the various maximal functions and pointwise convergence results concerning layer potentials that follow when their result is combined with arguments found in the paper of Fabes, Jodeit, and Rivi  re. The double layer potential for a function, f , defined on ∂D is defined by

$$\mathcal{K}f(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle Q - X, N(Q) \rangle}{|Q - X|^n} f(Q) dQ, \quad X \in \mathbf{R}^n \setminus \partial D,$$

where ω_n equals the surface measure of the unit sphere in \mathbf{R}^n , $dQ = d\sigma(Q)$, and $N(Q)$ denotes the outer unit normal to D . The single layer potential is defined, for $n > 2$, by

$$Sf(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{f(Q)}{|Q - X|^{n-2}} dQ, \quad X \in \mathbf{R}^n,$$

and, for $n = 2$, by

$$Sf(X) = \frac{1}{2\pi} \int_{\partial D} \log |Q - X| f(Q) dQ, \quad X \in \mathbf{R}^2.$$

$\mathcal{K}f$ and Sf are harmonic functions in $\mathbf{R}^n \setminus \partial D$. For $P \in \partial D$ the boundary layer potential and its adjoint are defined by

$$\begin{aligned}
 Kf(P) &= \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} f(Q) dQ \\
 &\equiv \lim_{\varepsilon \rightarrow 0} \frac{1}{\omega_n} \int_{|P - Q| > \varepsilon} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} f(Q) dQ
 \end{aligned}$$

and

$$K^*f(P) = \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P - Q, N(P) \rangle}{|Q - P|^n} f(Q) dQ.$$

In Section 2 we establish two fundamental operator inequalities,

$$\|(\tfrac{1}{2}I \mp K^*)f\|_2 \leq C \left(\|(\tfrac{1}{2}I \pm K^*)f\|_2 + \left| \int_{\partial D} Sf d\sigma \right| \right), \text{ for all } f \in L^2(\partial D).$$

In Section 3 the above inequalities are used to resolve basic functional analytic questions such as the closure and denseness of the range of $\frac{1}{2}I + K^*$ on $L^2(\partial D)$. In effect the inequalities allow one to circumvent the fact that, unlike on C^1 domains, the operator, K , is not compact. Additional potential theoretic arguments lead us to conclude that the double layer boundary potential, $\frac{1}{2}I + K$, is invertible on $L^2(\partial D)$. The operator $\frac{1}{2}I - K^*$ is shown to be invertible on the functions with mean value zero in $L^2(\partial D)$. The single layer potential itself is shown to be invertible from $L^2(\partial D)$ to $L^2_1(\partial D)$ and in addition $\frac{1}{2}I + K$ is shown to be invertible from $L^2_1(\partial D)$ to $L^2_1(\partial D)$. Also included in this section are two uniqueness theorems.

The special case of bounded Lipschitz domains in the plane is considered in Section 4 and invertibility questions for the potentials are completely resolved. In particular it is shown that $\frac{1}{2}I + K$ is invertible on $L^p(\partial D)$, $2 \leq p < \infty$, and that $\frac{1}{2}I - K^*$ is invertible on the $L^p(\partial D)$ functions with mean value zero, $1 < p \leq 2$.

Section 5 contains the regularity result for the Dirichlet problem when $n \geq 3$ and $1 < p \leq 2$.

Section 0 contains definitions concerning nontangential cones, coordinate cylinders, and maximal functions. It also contains the statements of three important results of B. E. J. Dahlberg.

Let us here set down some other conventions that will be used throughout the paper. The subspace of $L^p(\partial D)$ functions with mean value zero will be denoted by $L^p_0(\partial D)$. If the domain of integration is clear, L^p norms will be written simply as $\|\cdot\|_p$. For $1 \leq p \leq \infty$ the extended real number, p' , is defined by $1/p + 1/p' = 1$. Given a point, Q , on the boundary of any Lipschitz domain, $N(Q)$ will always denote the outer unit normal to the domain at Q if it exists. The phrases "almost everywhere" or "almost every" (a.e.) will be taken to mean with respect to surface measure. Thus $N(Q)$ exists a.e. on ∂D . The inner product in \mathbf{R}^n will be written $\langle \cdot, \cdot \rangle$ and the

Lebesgue (surface) measures of sets will be denoted by $|\cdot|$. An open ball of radius r and center X in \mathbf{R}^n be denoted by $B(X; r) = \{Y: |X - Y| < r\}$. The support of a function, φ , will be denoted $\text{supp}(\varphi)$. In general the letter C will stand for constants that depend only on the Lipschitz nature of the domain, D , and the L^p space under consideration. The Laplace operator, $\sum_{k=1}^n \partial^2 / \partial x_k^2$, will be denoted by Δ .

This paper is a revision of my thesis written at The University of Minnesota. My advisor was Eugene B. Fabes, without whose advice and encouragement the work here would not have begun nor could have come to such a happy conclusion.

I would also like to thank Max Jodeit, Jr., and Carlos Kenig for valuable comments and suggestions.

0. SOME DEFINITIONS AND KNOWN RESULTS

0.0. This section is devoted to establishing the notations for certain geometrical objects related to a Lipschitz domain. Notations for non-tangential maximal functions are also established. These notations will be used throughout the paper. In addition three theorems of B. E. J. Dahlberg are stated.

0.1. Let $D \subset \mathbf{D}^n$, $n \geq 2$, be a bounded Lipschitz domain with boundary, ∂D . See Section 0.5 below.

By a cylinder, $Z(X, r)$, we mean an open, right circular, doubly truncated cylinder centered at $X \in \mathbf{R}^n$ with radius equal to r . A coordinate cylinder, $Z = Z(Q, r)$, $Q \in \partial D$, will be defined by the following properties.

- (i) The bases of Z are some positive distance from ∂D .
- (ii) There is a rectangular coordinate system for \mathbf{R}^n , (x, s) , $x \in \mathbf{R}^{n-1}$, $s \in \mathbf{R}$, with s -axis containing the axis of Z .
- (iii) There is an associated function $\varphi = \varphi_Z: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ that is Lipschitz, i.e., $|\varphi(x) - \varphi(y)| \leq C|x - y|$, $C = C_Z < \infty$, for all $x, y \in \mathbf{R}^{n-1}$.
- (iv) $Z \cap D = Z \cap \{(x, s): s > \varphi(x)\}$.
- (v) $Q = (0, \varphi(0))$.

The pair (Z, φ) will be called a coordinate pair. In addition we will often write $Z_e = Z \setminus \bar{D}$ and $Z_i = Z \cap D$ and call these the exterior and interior cylinders, respectively. For any positive number, ν , $\nu Z(Q, r)$ will denote the cylinder $\{X \in \mathbf{R}^n: Q + (X - Q)/\nu \in Z\}$, i.e., the dilation of Z about Q by a factor of ν .

0.2. By compactness it is possible to cover ∂D with a finite number of coordinate cylinders Z_1, \dots, Z_N . However, it is possible and often

convenient to do this in such a way that for each Z_j there is a coordinate pair (Z_j^*, φ_j) with $Z_j^* = v_j Z_j$, where v_j is some sufficiently large positive number. For example, it can be useful to think of $v_j > 10(1 + \|\nabla \varphi_j\|_{L^\infty(\mathbf{R}^{n-1})})^{1/2}$.

Whenever we cover ∂D with coordinate cylinders we will assume also that the coordinate cylinders Z^* exist. Note that φ_j may be taken to have compact support in \mathbf{R}^{n-1} .

For a Lipschitz domain, D , there are numbers, $M < \infty$, so that for any covering of coordinate cylinders the φ_j all have Lipschitz norm less than or equal to M . The smallest such number is called the Lipschitz constant for D .

0.3. We remark that given $\varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$, Lipschitz with compact support, there is a sequence of $\psi_j \in C_0^\infty(\mathbf{R}^{n-1})$ such that $\psi_j \rightarrow \varphi$ uniformly, $\nabla \psi_j \rightarrow \nabla \varphi$ in every $L^q(\mathbf{R}^{n-1})$, $1 \leq q < \infty$, and $\|\nabla \psi_j\|_\infty$ is uniformly bounded. This may be effected by convolving φ with a smooth approximation to the identity in \mathbf{R}^{n-1} and using the L^q modulus of continuity of $\nabla \varphi$. Thus whenever φ is such a function and we write $\psi_j \rightarrow \varphi$ we will mean that the convergence is in the sense described in this paragraph.

0.4. By a cone we mean an open, circular, doubly truncated cone with two non-empty, convex components. If $Q \in \partial D$, $\Gamma(Q)$ will denote a cone with vertex at Q and one component in D and the other in $\mathbf{R}^n \setminus \bar{D}$. The component interior to D will be denoted by $\Gamma_i(Q)$ and the component exterior to \bar{D} will be denoted by $\Gamma_e(Q)$.

Assigning one cone, $\Gamma(Q)$, to each $Q \in \partial D$, we call the resulting family, $\{\Gamma(Q): Q \in \partial D\}$, regular if there is a finite covering of ∂D by coordinate cylinders, as described above, such that for each $(Z(P, r), \varphi)$ there are three cones, α , β , and γ , each with vertex at the origin and axis along the axis of Z such that

$$\alpha \subset \bar{\beta} \setminus \{0\} \subset \gamma$$

and for all $(x, \varphi(x)) = Q \in \frac{4}{3}Z^* \cap \partial D$

$$\alpha + Q \subset \Gamma(Q) \subset \overline{\Gamma(Q)} \setminus \{Q\} \subset \beta + Q,$$

$$(\gamma + Q)_i \subset D \cap Z^*, \quad \text{and} \quad (\gamma + Q)_e \subset Z^* \setminus \bar{D}.$$

See [8, p. 298] and the remarks preceding Theorem 1.12.

0.5. DEFINITION OF LIPSCHITZ DOMAIN. An open set $D \subset \mathbf{R}^n$ is called a Lipschitz domain if for each $Q \in \partial D$ there exist a rectangular coordinate system, (x, s) , $x \in \mathbf{R}^{n-1}$, $s \in \mathbf{R}$, a neighborhood, $U(Q) \equiv U \subset \mathbf{R}^n$ containing Q , and a function $\varphi_Q \equiv \varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that

- (i) $|\varphi(x) - \varphi(y)| \leq C_Q |x - y|$ for all $x, y \in \mathbf{R}^{n-1}$, $C_Q < \infty$;
- (ii) $U \cap D = \{(x, s): s > \varphi(x)\} \cap U$.

The coordinate systems, (x, s) , may always be taken to be a rotation and translation of the standard rectangular coordinates for \mathbf{R}^n .

0.6. Depending on where a given function, u , is defined we will write

$$u^*(Q) = N(u, \Gamma)(Q) = \sup_{\Gamma(Q)} |u(X)|,$$

$$u_i^*(Q) = N(u, \Gamma_i)(Q) = \sup_{\Gamma_i(Q)} |u(X)|,$$

and

$$u_e^*(Q) = N(u, \Gamma_e)(Q) = \sup_{\Gamma_e(Q)} |u(X)|$$

for nontangential maximal functions. The Hardy–Littlewood maximal function of a function f defined on ∂D or \mathbf{R}^{n-1} will be denoted by $\mathcal{M}f$. Thus for $f \in L^p(\partial D)$, $1 \leq p \leq \infty$,

$$\mathcal{M}f(Q) = \sup_{Q \in Z \cap \partial D} \frac{1}{\sigma(Z \cap \partial D)} \int_{Z \cap \partial D} |f| d\sigma,$$

where Z ranges over all coordinate cylinders for D containing Q .

0.7. Given a family of regular cones for $D \subset \mathbf{R}^n$, $\{\Gamma\}$, $n \geq 2$, define the interior and exterior Lusin area integrals for a function, u , by

$$A(u, \Gamma_i)(P) = \left(\int_{\Gamma_i(P)} |\nabla u(X)|^2 |X - P|^{2-n} dX \right)^{1/2}$$

and

$$A(u, \Gamma_e)(P) = \left(\int_{\Gamma_e(P)} |\nabla u(X)|^2 |X - P|^{2-n} dX \right)^{1/2}.$$

The reader will note that many of the arguments of Section 5 owe their conception to results of B. E. J. Dahlberg. We list three of these.

THEOREM 0.8D [8]. *Let D be a bounded connected Lipschitz domain in \mathbf{R}^n , $n \geq 2$, and $\{\Gamma\}$ a family of regular cones for D . Let P^* denote a fixed point of D . Let u be harmonic in D and vanishing at P^* . Then*

$$\|(u)_i^*\|_{L^p(\partial D)} \leq C \|A(u, \Gamma_i)\|_{L^p(\partial D)} \leq C' \|(u)_i^*\|_{L^p(\partial D)}$$

for $0 < p < \infty$, where the constants depend on D , $\{\Gamma\}$, p , and P^* .

THEOREM 0.9D [7]. *With D and $\{\Gamma\}$ as in Theorem 0.8D and $f \in L^p(\partial D)$, $2 \leq p \leq \infty$, there is a unique harmonic function, u , defined in D such that*

$$(i) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma_i(P)}} u(X) = f(P) \quad \text{a.e.}$$

and

$$(ii) \quad \|(u)_i^*\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)},$$

where C depends only on D , $\{\Gamma\}$, and p .

THEOREM 0.10D [6]. *With D as in Theorem 0.8D let ω denote harmonic measure with respect to some fixed point in D . Then on ∂D , $d\omega = k d\sigma$, where $k \in L^2(\partial D; d\sigma)$.*

1. TRACE THEOREMS FOR POTENTIALS

1.0. In this section we state without proof several classical formulas involving the boundary values of layer potentials. These are justified on Lipschitz domains by the recent and celebrated result of Coifman, McIntosh, and Meyer [5], which together with the method of rotations of A. P. Calderón [1] allows one to produce patterns of arguments like those found in [9] for C^1 domains. In particular the classical jump relations obeyed by the double layer potential (Theorem 1.10) and by the normal derivative of the single layer potential (Theorem 1.11) may be established. These are to be contrasted with the continuity across the boundary enjoyed by the tangential derivatives of the single layer potential (Theorem 1.6) and by the operators of Lemma 1.5. See also [3, pp. 258–261].

1.1. Notation. Throughout this section $D \subset \mathbf{R}^n$ will denote a bounded Lipschitz domain with a fixed regular family of cones, $\{\Gamma(P): P \in \partial D\} \equiv \{\Gamma\}$. The letters P and Q will denote points on ∂D and the letter X will denote points in $\mathbf{R}^n \setminus \partial D$. It will be assumed that $1 < p < \infty$ and the letter C will denote a constant that will depend at most on p , ∂D , and $\{\Gamma\}$.

The first two lemmas justify the application of Lebesgue's dominated convergence theorem in many situations throughout this paper.

LEMMA 1.2. *Let*

$$A_* f(P) = \sup_{\epsilon \rightarrow 0} \left| \int_{|Q-P| > \epsilon, Q \in \partial D} \frac{P-Q}{|P-Q|^n} f(Q) dQ \right|, \quad P \in \partial D.$$

Then $\|A_ f\|_p \leq C \|f\|_p$.*

Recall that

$$\nabla S f(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{X - Q}{|X - Q|^n} f(Q) dQ, \quad X \in \mathbf{R}^n \setminus \partial D.$$

LEMMA 1.3. $\|(\nabla S f)^*\|_p \leq C \|f\|_p$.

1.4. DEFINITION. Consider a coordinate pair, $(Z(Q_0, r), \varphi)$. Define for almost every $P = (x, \varphi(x)) \in Z^* \cap \partial D$ the vectors

$$T_l(P) = (0, \dots, 0, -1, 0, \dots, 0, -D_l \varphi(x)) / (1 + |\nabla \varphi(x)|^2)^{1/2},$$

where -1 appears in the l th place, $l = 1, \dots, n-1$. For almost every $P \in Z^* \cap \partial D$ these vectors are tangent to ∂D , linearly independent, and uniformly bounded in norm away from 0 and ∞ .

LEMMA 1.5. With Z and φ as described above let $T(Q)$ be one of the $T_l(Q)$ and take $\text{supp}(f) \subset \bar{Z} \cap \partial D$. If $f \in L^p(\partial D)$, then

$$(i) \quad \left(S \frac{\partial}{\partial T} \right) f(P) \equiv \lim_{\varepsilon \rightarrow 0} \frac{1}{\omega_n} \int_{|P-Q| > \varepsilon} \frac{\langle P-Q, T(Q) \rangle}{|P-Q|^n} f(Q) dQ$$

exists in $L^p(\partial D)$ and pointwise a.e., and

$$(ii) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X-Q, T(Q) \rangle}{|X-Q|^n} f(Q) dQ = \left(S \frac{\partial}{\partial T} \right) f(P)$$

for almost all $P \in \partial D$.

Note that the approach in (ii) is in both the interior and exterior components of the cone $\Gamma(P)$.

THEOREM 1.6. With Z , φ , and $T(Q)$ as in the hypotheses to Lemma 1.5 take any $f \in L^p(\partial D)$. Then

$$(i) \quad \frac{\partial}{\partial T} S f(P) \equiv \lim_{\varepsilon \rightarrow 0} \frac{1}{\omega_n} \int_{|P-Q| > \varepsilon} \frac{\langle P-Q, T(P) \rangle}{|P-Q|^n} f(Q) dQ$$

exists in $L^p(\partial D \cap Z)$ and pointwise almost everywhere in $\partial D \cap Z$, and

$$(ii) \quad \begin{aligned} \lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \langle T(P), \nabla S f(X) \rangle &= \lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X-Q, T(P) \rangle}{|X-Q|^n} f(Q) dQ \\ &= \frac{\partial}{\partial T} S f(P) \end{aligned}$$

for almost every $P \in \partial D \cap Z$.

1.7. DEFINITION. For a bounded Lipschitz domain $D \subset \mathbf{R}^n$ we say that $f \in L_1^p(\partial D)$ if $f \in L^p(\partial D)$ and if for each coordinate pair (Z, φ) there are $L^p(Z \cap \partial D)$ functions g_1, \dots, g_{n-1} so that

$$\int_{\mathbf{R}^{n-1}} h(x) g_i(x, \varphi(x)) dx = \int_{\mathbf{R}^{n-1}} D_{x_i} h(x) f(x, \varphi(x)) dx$$

holds for all $h \in C_0^\infty(Z \cap \mathbf{R}^{n-1})$.

Fixing a covering of ∂D by cylinders Z_1, \dots, Z_N , $f \in L_1^p(\partial D)$ may be normed by the sum of the L^p norms of all the locally defined g_i together with the L^p norm of f .

LEMMA 1.8. $S: L^p(\partial D) \rightarrow L_1^p(\partial D)$ is a bounded operator and for a.e. $P \in \partial D$

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma_i(P)}} Sf(X) = \lim_{\substack{X \rightarrow P \\ X \in \Gamma_e(P)}} Sf(X).$$

1.9. DEFINITION. Given an $f \in L_1^p(\partial D)$ it is possible to define a unique vector, $\nabla_t f(P) \in \mathbf{R}^n$, at almost every $P \in \partial D$ so that $\|f\|_{L^p(\partial D)} + \|\nabla_t f\|_{L^p(\partial D)}$ is equivalent to the norm of Definition 1.7. The resulting vector field, $\nabla_t f$, will be called the tangential gradient of f . In local coordinates $-\nabla_t f$ may be realized as

$$(g_1(x, \varphi(x)), \dots, g_{n-1}(x, \varphi(x)), 0) \\ - \langle (g_1(x, \varphi(x)), \dots, g_{n-1}(x, \varphi(x)), 0), N(x, \varphi(x)) \rangle N(x, \varphi(x)).$$

That $\nabla_t f$ is independent of the coordinate systems used to define it may be shown in a variety of ways. If f is a differentiable function in \mathbf{R}^n and $P \in \partial D$, then $\nabla f(P) = \nabla_t f(P) + \langle N(P), \nabla f(P) \rangle N(P)$.

Thus $L_1^p(\partial D)$ may be normed by

$$\|f\|_{L_1^p} = \|f\|_p + \|\nabla_t f\|_p.$$

Recall the definition of $\mathcal{K}f(X)$ from the Introduction.

THEOREM 1.10.

$$(i) \quad \lim_{\epsilon \rightarrow 0} K_\epsilon f(P) \equiv \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} f(Q) dQ \equiv Kf(P)$$

exists in $L^p(\partial D)$ and pointwise for a.e. $P \in \partial D$, and

$$(ii) \quad \lim_{X \rightarrow P} \mathcal{K}f(X) = (\tfrac{1}{2}I + K)f(P), \quad X \in \Gamma_i(P) \\ = -(\tfrac{1}{2}I - K)f(P), \quad X \in \Gamma_e(P)$$

for almost every $P \in \partial D$.

THEOREM 1.11.

$$(i) \quad \lim_{\varepsilon \rightarrow 0} K_\varepsilon^* f(P) = \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P - Q, N(P) \rangle}{|P - Q|^n} f(Q) dQ \equiv K^* f(P)$$

exists in $L^p(\partial D)$ and pointwise a.e., and

$$(ii) \quad \begin{aligned} \lim_{X \rightarrow P} \langle N(P), \nabla S f(X) \rangle &= -(\tfrac{1}{2}I - K^*)f(P), & X \in \Gamma_i(P) \\ &= (\tfrac{1}{2}I + K^*)f(P), & X \in \Gamma_e(P) \end{aligned}$$

for a.e. $P \in \partial D$.

It will be necessary to approximate a given Lipschitz domain, D , by sequences of C^∞ domains, $\Omega_j, j = 1, 2, \dots$, in the manner described in the next theorem. For verification the reader may consult [18, 19] or [21].

THEOREM 1.12. *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Then the following propositions hold.*

(i) *There is a regular family of cones $\{\Gamma\}$ for D as described in Section 0.4 above.*

(ii) *There is a sequence of C^∞ domains, $\Omega_j \subset D$, and homeomorphisms, $\Lambda_j: \partial D \rightarrow \partial \Omega_j$, such that $\sup_{Q \in \partial D} |Q - \Lambda_j(Q)| \rightarrow 0$ as $j \rightarrow \infty$ and for all j and all $Q \in \partial D$ $\Lambda_j(Q) \in \Gamma_i(Q)$.*

(iii) *There is a covering of ∂D by coordinate cylinders, Z , so that given a coordinate pair, (Z, φ) , then $Z^* \cap \partial \Omega_j$ is given for each j as the graph of a C^∞ function φ_j such that $\varphi_j \rightarrow \varphi$ uniformly, $\|\nabla \varphi_j\| \leq \|\nabla \varphi\|_\infty$, and $\nabla \varphi_j \rightarrow \nabla \varphi$ pointwise a.e. and in every $L^q(Z^* \cap \mathbf{R}^{n-1})$, $1 \leq q < \infty$.*

(iv) *There are positive functions $\omega_j: \partial D \rightarrow \mathbf{R}_+$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial D$, $\int_E \omega_j d\sigma = \int_{\Lambda_j(E)} d\sigma_j$, and so that $\omega_j \rightarrow 1$ pointwise a.e. and in every $L^q(\partial D)$, $1 \leq q < \infty$.*

(v) *The normal vectors to Ω_j , $N(\Lambda_j(Q))$, converge pointwise a.e. and in every $L^q(\partial D)$, $1 \leq q < \infty$, to $N(Q)$. An analogous statement holds for locally defined tangent vectors.*

(vi) *There exist C^∞ vector fields, \mathbf{h} , in \mathbf{R}^n such that for all j and $Q \in \partial D$ $\langle \mathbf{h}(\Lambda_j(Q)), N(\Lambda_j(Q)) \rangle \geq C > 0$, where C depends only on \mathbf{h} and the Lipschitz constant for D .*

1.13. DEFINITION. The approximation scheme comprising (ii)–(iv) of the theorem will be denoted $\Omega_j \uparrow D$. The notation $\Omega_j \downarrow D$ will refer to the similar scheme when $\Omega_j \supset D$.

It is convenient to write $A_j(Q) = Q_j$ and N^j or T^j for N or T when these denote normal or tangent vectors to Ω_j .

1.14. *Remark.* In the proof of Theorem 2.1, $f \in L^2(\partial D)$, $u(X) \equiv Sf(X)$, $\Omega_j \uparrow D$ and it is asserted that

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^2(\partial \Omega_j)}^2 \equiv \int_{\partial \Omega_j} \left(\frac{\partial u}{\partial N^j} \right) d\sigma_j \rightarrow \int_{\partial D} \left[\left(\frac{1}{2}I - K^* \right) f \right]^2 d\sigma.$$

To see this the left side is written as

$$\begin{aligned} & \int_{\partial D} \langle N^j(A_j(Q)) - N(Q), \nabla u(A_j(Q)) \rangle^2 \omega_j(Q) d\sigma(Q) \\ & + \int_{\partial D} \langle N(Q), \nabla u(A_j(Q)) \rangle^2 \omega_j(Q) d\sigma(Q). \end{aligned}$$

By Lemma 1.3 and (iv) of the last theorem the integrand of the first integral is dominated by $C[(\nabla Sf)^*]^2$. Therefore by (v) of the last theorem and dominated convergence the first integral converges to zero. Dominated convergence similarly applies to the second integral and Theorem 1.11 finally yields the assertion.

The above kind of argument illustrates the utility of the preceding convergence results, while used only implicitly in much of what follows.

2. AN OPERATOR INEQUALITY

The purpose of this section is to establish the following.

THEOREM 2.1. *Let D be a bounded Lipschitz domain in \mathbf{R}^n , $n \geq 3$, with connected boundary. Then for all $f \in L^2(\partial D)$*

- (i) $\|(\frac{1}{2}I - K^*)f\|_2 \leq C\{\|(\frac{1}{2}I + K^*)f\|_2 + |\int_{\partial D} Sf d\sigma|\}$, and
- (ii) $\|(\frac{1}{2}I + K^*)f\|_2 \leq C\{\|(\frac{1}{2}I - K^*)f\|_2 + |\int_{\partial D} Sf d\sigma|\}$,

where C depends only on the Lipschitz constant for D .

This theorem will be used repeatedly in the next section in order to prove the invertibility of boundary layer potentials on $L^2(\partial D)$. On a C^1 domain the operator, K , is compact on L^2 of the boundary. That is not the case for Lipschitz domains. A simple example of this may be found on page 110 of [10]. Theorem 2.1 serves as a substitute for compactness in the functional analytic arguments of Section 3. For example, the fact that $\frac{1}{2}I + K$ has closed range on L^2 follows easily from this theorem. See the proof of Theorem 3.1.

The proof of Theorem 2.1 is really a simple interpretation in terms of layer potentials of the following Rellich identity. See [19, p. 245] and also [13] or [14].

LEMMA 2.2. *Let Ω be a bounded C^∞ domain in \mathbf{R}^n , $n \geq 3$. Let u be a function such that either*

- (i) $u \in C^\infty(\bar{\Omega})$ and $\Delta u = 0$ in Ω , or
- (ii) $u \in C^\infty(\mathbf{R}^n \setminus \bar{\Omega})$, $\Delta u = 0$ in $\mathbf{R}^n \setminus \bar{\Omega}$, and $|u(x)| = O(|X|^{2-n})$ as $|X| \rightarrow \infty$.

Let \mathbf{h} be a C^∞ vector field in \mathbf{R}^n with compact support. Then

$$\begin{aligned} \int_{\partial\Omega} \langle N, \mathbf{h} \rangle |\nabla u|^2 d\sigma &= 2 \int_{\partial\Omega} \frac{\partial u}{\partial N} \langle \mathbf{h}, \nabla u \rangle d\sigma \\ &\quad + \int_{\Omega} \operatorname{div} \mathbf{h} |\nabla u|^2 - 2 \langle \nabla \mathbf{h}(\nabla u), \nabla u \rangle dX. \end{aligned}$$

Here $\nabla \mathbf{h}$ is an $n \times n$ matrix acting on ∇u .

A Poincaré-type inequality is also needed. The proof is standard. See, for example, [17].

LEMMA 2.3. *Let $\Omega_0 \subset \mathbf{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary. Let $\{\Omega_j\}$, $j = 1, 2, \dots$, be an approximating sequence of domains for Ω_0 in the sense of Definition 1.13. Then for all $j = 0, 1, \dots$, there is a constant, C , depending only on $\partial\Omega_0$ and p and independent of j such that for any $f \in L^p_0(\partial\Omega_j)$*

$$\int_{\partial\Omega_j} |f|^p d\sigma_j \leq C \int_{\partial\Omega_j} |\nabla_t f|^p d\sigma_j.$$

Proof of Theorem 2.1. We will prove part (i), part (ii) being similar.

Let $f \in L^2(\partial D)$ and put $u = Sf$. Let $\Omega_j \uparrow D$ and choose a C^∞ vector field \mathbf{h} so that on $\partial\Omega_j \langle N^j, \mathbf{h} \rangle \geq C > 0$, where C depends only on D . See Theorem 1.12. For any j write $\Omega = \Omega_j$. Note that on $\partial\Omega |\nabla u|^2 = |\nabla_t u|^2 + (\partial u / \partial N)^2$, where ∇_t denotes tangential differentiation to $\partial\Omega$. Also note that

$$\frac{\partial u}{\partial N} \langle \mathbf{h}, \nabla u \rangle = \left(\frac{\partial u}{\partial N} \right)^2 \langle \mathbf{h}, N \rangle + \frac{\partial u}{\partial N} \langle \alpha, \nabla u \rangle,$$

where α is tangent to $\partial\Omega$. See [13]. Applying Lemma 2.2 and the Schwarz inequality yields

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^2(\partial\Omega)}^2 \leq C \left\{ \|\nabla_t u\|_{L^2(\partial\Omega)}^2 + \|\nabla_t u\|_{L^2(\partial\Omega)} \left\| \frac{\partial u}{\partial N} \right\|_{L^2(\partial\Omega)} + \int_{\Omega} |\nabla u|^2 dX \right\}.$$

Since the last integral equals $\int_{\partial\Omega} u(\partial u/\partial N) d\sigma$ the Poincaré lemma, Schwarz, and a simple argument yield

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^2(\partial\Omega)} \leq C \|\nabla_t u\|_{L^2(\partial\Omega)},$$

where C depends only on D .

Letting $j \rightarrow \infty$ the comments of 1.14 apply to give

$$\|(\tfrac{1}{2}I - K^*)f\|_{L^2(\partial D)} \leq C \|\nabla_t Sf\|_{L^2(\partial D)}.$$

Part (i) will follow once it is shown by working outside D that

$$\|\nabla_t Sf\|_{L^2(\partial D)} \leq C \left\{ \left\| \left(\frac{1}{2}I + K^* \right) f \right\|_{L^2(\partial D)} + \left| \int_{\partial D} Sfd\sigma \right| \right\}$$

since by Theorem 1.6 the tangential derivatives of Sf are independent of interior or exterior approach. But this follows by utilizing Lemma 2.2 to bound ∇u in terms of $\partial u/\partial N$.

Note, however, that for $\Omega \supset D$, $\int_{\Omega} (\partial u/\partial N) d\sigma$ is not necessarily zero so that the use of the Poincaré lemma for

$$\int_{\partial\Omega} u \frac{\partial u}{\partial N} d\sigma = \int_{\partial\Omega} \left[u - |\partial\Omega|^{-1} \int_{\partial\Omega} u d\sigma \right] \frac{\partial u}{\partial N} d\sigma + |\partial\Omega|^{-1} \int_{\partial\Omega} u d\sigma \int_{\partial\Omega} \frac{\partial u}{\partial N} d\sigma$$

contributes the term $|\int_{\partial D} Sfd\sigma|$.

2.4. Remark. Lemma 2.2 shows that the results of [13, 14] on the Neumann problem and regularity for the Dirichlet problem extend to the more general Lipschitz domains in this paper.

3. INVERTIBILITY OF LAYER POTENTIALS IN L^2

3.0. In this section we prove the invertibility of the boundary layer potentials on various subspaces of $L^2(\partial D)$ when D is a bounded Lipschitz domain with connected boundary in \mathbf{R}^n , $n \geq 3$. The two-dimensional case will be considered in Section 4.

As an application of one of the results in this section we prove a uniqueness theorem for the Neumann problem for certain $p \leq 2$. We also state the uniqueness theorem for the Dirichlet problem.

Recall the classical method of layer potentials. The double layer potential $\mathcal{K}f(x)$ is harmonic in D and has nontangential boundary values $(\tfrac{1}{2}I + K)f$. Given boundary data, g , the Dirichlet problem can be solved if the integral equation, $(\tfrac{1}{2}I + K)f = g$, can be solved for f . Similarly the Neumann problem may be examined by considering $(\tfrac{1}{2}I - K^*)f = g$.

Of course for $2 \leq p \leq \infty$ existence and uniqueness of solutions for the L^p -Dirichlet problem have been established by Dahlberg. Also, Jerison and Kenig [15] have established existence and uniqueness for the L^2 -Neumann problem. The point of the next two theorems is that these solutions may be represented by potentials of $L^2(\partial D)$ functions.

Throughout this section the letter C will denote constants that depend only on the Lipschitz domain, D .

THEOREM 3.1. $\frac{1}{2}I + K: L^2(\partial D) \rightarrow L^2(\partial D)$ is invertible.

Proof. Let $(\frac{1}{2}I + K^*)f = 0$ a.e. Putting $u = Sf$, the comments of 1.14 justify

$$\int_{\mathbf{R}^n \setminus D} |\nabla u(X)|^2 dX = - \int_{\partial D} u \frac{\partial u}{\partial N} d\sigma.$$

The right side is zero by Theorem 1.11. Since $u(X) = O(|X|^{2-n})$ as $|X| \rightarrow \infty$ and $\mathbf{R}^n \setminus D$ is connected we conclude that Sf is zero outside D and therefore zero a.e. on ∂D . See [9]. Thus

$$\int_D |\nabla u(X)|^2 dX = \int_{\partial D} u \frac{\partial u}{\partial N} d\sigma = 0$$

since the single layer potential is continuous (a.e.) across the boundary. Consequently $Sf \equiv 0$ on \mathbf{R}^n , whence $(\frac{1}{2}I - K^*)f = 0$ a.e. Thus $f = 0$ and $\frac{1}{2}I + K$ has dense range on $L^2(\partial D)$.

By Banach's closed range theorem, to show that $\frac{1}{2}I + K$ is onto it suffices to show that $\frac{1}{2}I + K^*$ has closed range. Assume that $(\frac{1}{2}I + K^*)f_j \rightarrow g$ in $L^2(\partial D)$.

If $\|f_j\|_2 \leq B < \infty$ then one may say $f_j \rightarrow f$ weakly in $L^2(\partial D)$ and so for any $h \in L^2(\partial D)$

$$\begin{aligned} \int_{\partial D} gh &= \lim_j \int_{\partial D} (\frac{1}{2}I + K^*)f_j h = \lim_j \int_{\partial D} f_j (\frac{1}{2}I + K)h \\ &= \int_{\partial D} f (\frac{1}{2}I + K)h. \end{aligned}$$

Thus $(\frac{1}{2}I + K^*)f = g$. On the other hand if $\|f_j\|_2 \rightarrow \infty$, then after dividing by these norms we are reduced to

$$(\frac{1}{2}I + K^*)f_j \rightarrow 0 \quad \text{in } L^2(\partial D) \quad (1)$$

$$\|f_j\|_2 \equiv 1 \quad (2)$$

and thus arguing as above.

$$f_j \rightarrow f \quad \text{weakly in } L^2(\partial D),$$

whence $(\frac{1}{2}I + K^*)f = 0$ whence $f = 0$ and thus

$$f_j \rightarrow 0 \quad \text{weakly in } L^2(\partial D). \quad (3)$$

Statement (3) implies that $\int_{\partial D} S f_j \rightarrow 0$. Part (i) of Theorem 2.1 yields

$$\|(\frac{1}{2}I - K^*)f_j\|_2 \leq C \left(\|(\frac{1}{2}I + K^*)f_j\|_2 + \left| \int_{\partial D} S f_j \right| \right) \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

by (1) and the preceding sentence. Thus

$$\|f_j\|_2 \leq \|(\frac{1}{2}I + K^*)f_j\|_2 + \|(\frac{1}{2}I - K^*)f_j\|_2 \rightarrow 0,$$

contradicting (2). This shows that $\frac{1}{2}I + K$ is onto $L^2(\partial D)$.

Finally to show that $\frac{1}{2}I + K$ is one to one we show that $\frac{1}{2}I + K^*$ has dense range.

Let $\Omega_j \downarrow D$. Theorem 2.1 holds for each Ω_j with constants depending only on D . Recall now the homeomorphisms $A_j: \partial D \rightarrow \partial \Omega_j$, $A_j(Q) \equiv Q_j$ and the Jacobians ω_j from Theorem 1.12.

Take $g \in C_0^\infty(\mathbf{R}^n)$ and define the layer potentials K_j and S_j on $\partial \Omega_j$. Then (for example, see [9]) there are $f_j \in L^2(\partial \Omega_j)$ such that

$$(\frac{1}{2}I + K_j^*)f_j = g|_{\partial \Omega_j} \quad \text{a.e.}$$

Define on ∂D

$$F_j(Q) = f_j \circ A_j(Q) \omega_j(Q).$$

If $\|f_j\|_{L^2(\partial \Omega_j)} \leq B < \infty$, then the $\|F_j\|_{L^2(\partial D)}$ are also uniformly bounded. Thus $F_j \rightarrow F$ weakly in $L^2(\partial D)$.

Now take $h \in C^\infty(\mathbf{R}^n)$.

$$\begin{aligned} & \int_{\partial \Omega_j} (\frac{1}{2}I + K_j^*)f_j h \, d\sigma_j \\ &= \int_{\partial D} F_j(\frac{1}{2}I + K) h \, d\sigma + \int_{\partial D} F_j[(\frac{1}{2}I + K_j)h] \circ A_j - (\frac{1}{2}I + K)h \, d\sigma. \end{aligned}$$

Clearly the left side converges to $\int_{\partial D} g h \, d\sigma$. By weak convergence the first term on the right converges to $\int_{\partial D} (\frac{1}{2}I + K^*)F \cdot h \, d\sigma$ while the second converges to zero. This is because $\|((\frac{1}{2}I + K_j)h) \circ A_j - (\frac{1}{2}I + K)h\|_{L^2(\partial D)} \rightarrow 0$. To see this fix $\varepsilon > 0$ and write (recall that

$$\text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} dQ = \frac{1}{2}$$

or see [9]):

$$\begin{aligned}
 & ((\tfrac{1}{2}I + K_j)h) \circ A_j(P) - (\tfrac{1}{2}I + K)h(P) \\
 &= \frac{1}{\omega_n} \left\{ \int_{|Q_j - P_j| > \varepsilon} \frac{\langle Q_j - P_j, N(Q_j) \rangle}{|Q_j - P_j|^n} [h(Q_j) - h(P_j)] dQ_j \right. \\
 &\quad \left. - \int_{|Q - P| > \varepsilon} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} [h(Q) - h(P)] dQ \right\} \\
 &\quad + \frac{1}{\omega_n} \left\{ \int_{|Q_j - P_j| < \varepsilon} \frac{\langle Q_j - P_j, N(Q_j) \rangle}{|Q_j - P_j|^n} [h(Q_j) - h(P_j)] dQ_j \right. \\
 &\quad \left. - \int_{|Q - P| < \varepsilon} \frac{\langle Q - P, N(Q) \rangle}{|Q - P|^n} [h(Q) - h(P)] dQ \right\} + h(P_j) - h(P) \\
 &\equiv A_j(P) + B_j(P) + h(P_j) - h(P).
 \end{aligned}$$

The sequence of functions, $\{A_j\}$, on ∂D is a uniformly bounded sequence that converges pointwise to zero a.e. and therefore to zero in L^2 . Both integrals in B_j are for each P less than $\varepsilon C \|\nabla h\|_\infty$, where C depends only on D . Thus $\lim_j \|B_j\|_{L^2(\partial D)} \leq \varepsilon C \|\nabla h\|_{L^\infty(\mathbb{R}^n)}$. Since ε is arbitrary we conclude that $(\tfrac{1}{2}I + K^*)F = g|_{\partial D}$.

If $\|f_j\|_{L^2(\partial \Omega_j)} \rightarrow \infty$ we reduce to

$$\|f_j\|_{L^2(\partial \Omega_j)} \equiv 1 \quad (1')$$

$$\|(\tfrac{1}{2}I + K^*)f_j\|_{L^2(\partial \Omega_j)} \rightarrow 0 \quad (2')$$

and arguing as above

$$F_j \rightarrow 0 \quad \text{weakly in } L^2(\partial D). \quad (3')$$

Applying part (i) of Theorem 2.1 to each Ω_j will show (1'), (2'), and (3') to be contradictory once we establish

$$\left| \int_{\partial \Omega_j} S_j f_j d\sigma_j \right| \rightarrow 0.$$

One can see that this will be true by writing

$$\begin{aligned}
 (2-n)\omega_n \int_{\partial \Omega_j} S_j f_j(P_j) dP_j &= \int_{\partial D} dP \int_{\partial D} \frac{F_j(Q)}{|P-Q|^{n-2}} dQ \\
 &\quad + \int_{\partial D} (\omega_j(P) - 1) dP \int_{\partial D} \frac{F_j(Q)}{|P-Q|^{n-2}} dQ \\
 &\quad + \int_{\partial \Omega_j} dP_j \int_{\partial D} F_j(Q) \left[\frac{1}{|P_j - Q_j|^{n-2}} - \frac{1}{|P - Q|^{n-2}} \right] dQ.
 \end{aligned}$$

The first term converges to zero by (3'). The second converges to zero because the densities $\omega_j \rightarrow 1$ in L^q . For the third term fix $\varepsilon > 0$. Then it may be written

$$\begin{aligned} & \int_{\partial D} \omega_j(P) dP \int_{|P-Q|>\varepsilon} F_j(Q) \left[\frac{1}{|P_j-Q_j|^{n-2}} - \frac{1}{|P-Q|^{n-2}} \right] dQ \\ & + \int_{\partial D} \omega_j(P) dP \int_{|P-Q|<\varepsilon} F_j(Q) \left[\frac{1}{|P_j-Q_j|^{n-2}} - \frac{1}{|P-Q|^{n-2}} \right] dQ. \end{aligned}$$

The first term is dominated by

$$C \sup_{P \in \partial D} \left(\int_{|P-Q|>\varepsilon} \left| \frac{1}{(P_j-Q_j)^{n-2}} - \frac{1}{|P-Q|^{n-2}} \right|^2 dQ \right)^{1/2} \rightarrow 0.$$

The second is of order ε since we may integrate in P first. Thus

$$\lim_{j \rightarrow \infty} \left| \int_{\partial \Omega_j} S_j f_j d\sigma_j \right| \leq \varepsilon C.$$

But ε was arbitrary and the proof is complete.

COROLLARY 3.2. *Given data $g \in L^2(\partial D)$ the unique solution, u , to the Dirichlet Problem,*

- (i) $\Delta u = 0$ in D ,
- (ii) $u \rightarrow g$ a.e. in nontangential cones,
- (iii) $\|u^*\|_{L^2(\partial D)} < \infty$,

may be written $u(X) = \mathcal{K}(\frac{1}{2}I + K)^{-1}g(X)$.

The next theorem implies similar consequences for the Neumann problem and regularity for the Dirichlet problem. See Section 5.

- THEOREM 3.3.** (i) $\frac{1}{2}I - K^*: L_0^2(\partial D) \rightarrow L_0^2(\partial D)$,
 (ii) $S: L^2(\partial D) \rightarrow L_1^2(\partial D)$, and
 (iii) $\frac{1}{2}I + K: L_1^2(\partial D) \rightarrow L_1^2(\partial D)$

are invertible operators.

Proof. To prove (i) first note that $\frac{1}{2}I - K^*$ maps all of $L^2(\partial D)$ into $L_0^2(\partial D)$ since $\int_{\partial D} (\partial/\partial N) S f d\sigma = 0$ by the Gauss divergence theorem.

Now assume $(\frac{1}{2}I - K^*)f = 0$ a.e. and $\int_{\partial D} f = 0$. Then as in the proof of Theorem 3.1, $\int_D |\nabla S f(X)|^2 dX = 0$ so that $S f \equiv \text{constant}$ in D . But the assumptions on f imply that $\int_{\partial D} (\frac{1}{2}I + K^*)f d\sigma = 0$ also. Thus

$\int_{\mathbf{R}^n \setminus D} |\nabla S f(X)|^2 dX = \text{constant} \cdot \int_{\partial D} (\frac{1}{2}I + K^*) f = 0$, whence by the decay of the single layer potential at infinity $S f \equiv 0$ in $\mathbf{R}^n \setminus D$ and thus in all of \mathbf{R}^n . Thus $f = 0$ and $\frac{1}{2}I - K^*$ is one to one on $L_0^2(\partial D)$.

That $\frac{1}{2}I - K^*$ has closed range follows from arguments like those used in the proof of Theorem 3.1.

To show that $\frac{1}{2}I - K^*$ has dense range in L_0^2 let $g \in C^\infty(\mathbf{R}^n)$ so that $\int_{\partial D} g d\sigma = 0$ and put for $\Omega_j \downarrow D$ $m_j = (1/|\partial\Omega_j|) \int_{\partial\Omega_j} g d\sigma_j$. Now solve $(\frac{1}{2}I - K_j^*) f_j = g|_{\partial\Omega_j} - m_j$ for $f_j \in L_0^2(\partial\Omega_j)$. Putting, as in the proof of Theorem 3.1,

$$F_j = (f_j \circ A_j) \omega_j$$

we have $F_j \in L_0^2(\partial D)$. Note also that $m_j \rightarrow 0$ as $j \rightarrow \infty$. Now the arguments used in the proof of Theorem 3.1 go through with trivial modifications to complete the proof of (i).

To prove (ii) it is not hard to see that $S f = 0$ a.e. on ∂D implies $\int_{\mathbf{R}^n} |\nabla S f|^2 = 0$ and thus $f = 0$ so that S is one to one.

Since $\frac{1}{2}I - K^*: L^2 \rightarrow L_0^2$ and is invertible on the latter, there exists a unique function, f_0 , in the kernel of $\frac{1}{2}I - K^*$ such that $\int_{\partial D} f_0 d\sigma = 1$. $S f_0$ is constant in D and the constant is not equal to zero since S is one to one.

Now let $g \in L_1^2(\partial D)$ and denote the normal derivative of its Poisson extension by Ag . By [14] and Remark 2.4, $\|Ag\|_{L^2(\partial D)} \leq C \|g\|_{L_1^2(\partial D)}$. Since $-(\partial/\partial N) S(\frac{1}{2}I - K^*)^{-1} Ag = Ag$ uniqueness for the Neumann problem (see [13, 14]) implies that $S(\frac{1}{2}I - K^*)^{-1} Ag$ differs from the Poisson extension of g by a constant. But constants are in the range of S . This proves (ii).

To prove (iii) let $g \in L_1^2(\partial D)$ and put $v(X) = S(\frac{1}{2}I + K^*)^{-1} S^{-1} g(X)$. Then v has nontangential boundary values $f \in L_1^2(\partial D)$. Also

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma_\rho(P)}} N_P \cdot \nabla v(X) = S^{-1} g(P) \quad \text{a.e.}$$

Since v decays rapidly enough at ∞ when $n \geq 3$ we may write the Green's formula for the exterior of D

$$\begin{aligned} v(X) = & -\frac{1}{\omega_n} \int_{\partial D} \frac{\langle Q - X, N(Q) \rangle}{|Q - X|^n} f(Q) dQ \\ & - \frac{1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X - Q|^{n-2}} S^{-1} g(Q) dQ. \end{aligned}$$

Going to the boundary we obtain

$$f = (\frac{1}{2}I - K) f + g.$$

Thus $\frac{1}{2}I + K$ maps onto L_1^2 . Since any $f \in L_1^2$ may be written $S(\frac{1}{2}I + K^*)^{-1}S^{-1}g$ for some $g \in L_1^2$ we also see that $\frac{1}{2}I + K$ is into L_1^2 from L_1^2 . We already know that $\frac{1}{2}I + K$ is one to one. The formula

$$S(\frac{1}{2}I + K^*)S^{-1} = \frac{1}{2}I + K$$

which we have established in the course of the proof shows that $\frac{1}{2}I + K: L_1^2 \rightarrow L_1^2$ is bounded.

COROLLARY 3.4. *Given data $g \in L_0^2(\partial D)$ the unique solution, u , to the Neumann problem,*

- (i) $\Delta u = 0$ in D ,
- (ii) $\partial u / \partial N = g$ a.e. on ∂D ,
- (iii) $\int_{\partial D} u f_0 d\sigma = 0$,
- (iv) $\|(\nabla u)^*\|_{L^2(\partial D)} < \infty$,

may be written

$$u(X) = -S(\frac{1}{2}I - K^*)^{-1}g(X).$$

Here f_0 is the function mentioned in the proof of Theorem 3.3.

COROLLARY 3.5. *Let $g \in L_1^2(\partial D)$ be given.*

- (i) *There is a unique $f \in L_1^2(\partial D)$ so that $Sf = g$ a.e. on ∂D and $\|(\nabla Sf)_i^*\|_2 < \infty$.*
- (ii) *There is a unique $h \in L_1^2(\partial D)$ so that $\mathcal{K}h = g$ a.e. on ∂D and $\|(\nabla \mathcal{K}h)_i^*\|_2 < \infty$.*

An interesting consequence of Theorem 3.3 is the following uniqueness result for the L^p -Neumann problem.

LEMMA 3.6. *Let $D \subset \mathbb{R}^n$, $n \geq 2$. Let $1 < p \leq 2$ be such that $L_1^p(\partial D)$ embeds in $L^2(\partial D)$ by the Sobolov lemma, e.g., for $n \geq 3$, $p \geq 2((n-1)/(n+1))$. Let $\{\Gamma\}$ be a regular family of cones for D . If*

- (i) $\Delta u = 0$ in D ,
- (ii) $\lim_{X \rightarrow P, X \in \Gamma_i(P)} \langle N(P), \nabla u(X) \rangle = 0$ a.e., and
- (iii) $(\nabla u)_i^* \in L^p(\partial D)$

then $u \equiv \text{constant}$ in D .

Proof. Let $\Omega_j \uparrow D$. For $X \in D$ and j large enough and $n \geq 3$

$$u(X) = \frac{1}{\omega_n} \int_{\partial\Omega_j} \frac{\langle N(Q), Q - X \rangle}{|Q - X|^n} u(Q) dQ \\ - \frac{1}{\omega_n(2-n)} \int_{\partial\Omega_j} \frac{1}{|Q - X|^{n-2}} \frac{\partial u}{\partial N}(Q) dQ. \quad (1)$$

If $Y \in D$ is close to ∂D let $E = \{P \in \partial D: \Gamma_i(P) \ni Y\}$. For each $P \in E$ let $A(P)$ be the point on the base of $\Gamma_i(P)$ that also is on the line containing Y and P . Then

$$u(Y) = u(A(P)) + \int_0^1 \langle \nabla u(tY + (1-t)A(P)), (Y - A(P)) \rangle dt.$$

Letting \mathcal{M} denote the Hardy–Littlewood maximal function, an integration over E yields

$$|u(Y)| \leq C \left(\sup_{\substack{X \in D \\ X \notin \bigcup_{Q \in \partial D} \Gamma_i(Q)}} |u(X)| + \mathcal{M}((\nabla u)_i^*)(P_0) \right)$$

for any $P_0 \in E$. Thus $(u)_i^* \in L^p(\partial D)$ and consequently we may assume $u|_{\partial\Omega_j} \rightarrow f \in L^p(\partial D)$ weakly. Thus (1) becomes $u(X) = \mathcal{R}f(X)$ as $j \rightarrow \infty$ by the last remark and hypothesis (ii). But then $u|_{\partial\Omega_j} \rightarrow (\frac{1}{2}I + K)f$ in L^p norm, whence we must have

$$f = (\tfrac{1}{2}I + K)f \text{ a.e., i.e., } (\tfrac{1}{2}I - K)f = 0. \quad (2)$$

The above type of analysis goes through for $n = 2$ as well. Now if $f \in L^2(\partial D)$, then by (i) of Theorem 3.3, f must be constant, which implies that u is constant. But it is true that $f \in L^2(\partial D)$ since $f \in L_1^p(\partial D)$. For if $h \in C_0^\infty(\mathbf{R}^{n-1} \cap Z)$ then

$$\int_{\mathbf{R}^{n-1}} D_{x_i} h(x) f(x, \varphi(x)) dx = \lim_{t \downarrow 0} \int_{\mathbf{R}^{n-1}} D_{x_i} h(x) u(x, t + \varphi(x)) dx \\ = \lim_{t \downarrow 0} \int_{\mathbf{R}^{n-1}} h(x) \langle T_i(x, \varphi(x)), \nabla u(x, t + \varphi(x)) \rangle (1 + |\nabla \varphi(x)|^2)^{1/2} dx.$$

But again we may assume that the inner product converges weakly to an $L^p(\partial D)$ function since $(\nabla u)_i^* \in L^p(\partial D)$. Thus $f \in L_1^p(\partial D)$ and the proof is complete for $n \geq 3$. That the proof holds for $n = 2$ follows from results in Section 4.

Let us also take this opportunity to state a uniqueness result for the Dirichlet problem. The proof may be found in [9, pp. 183–184].

LEMMA 3.7. *With D and $\{\Gamma\}$ as in Lemma 3.6 let*

- (i) $\Delta u = 0$ in D ,
- (ii) $\lim_{X \rightarrow P, X \in \Gamma_i(P)} u(X) = 0$ a.e.,
- (iii) $\|(u)_i^*\|_{L^2(\partial D)} < \infty$;

then $u \equiv 0$ in D .

We remark that Dahlberg's work on the Green's function, [6], justifies the arguments of [9] on Lipschitz domains.

4. INVERTIBILITY OF LAYER POTENTIALS IN L^p FOR LIPSCHITZ DOMAINS IN THE PLANE

4.0. In this section we show that the layer potentials are invertible on various subspaces of $L^p(\partial D)$ for general p , $1 < p < \infty$, when the Lipschitz domain, D , is a bounded subset of the plane with connected boundary. After first establishing L^2 results as in Section 3, we use the Lusin area integral to establish two operator inequalities (Lemma 4.8) that serve us here as the inequalities of Theorem 2.1 served us in the last section. It follows that $\frac{1}{2}I + K$ is invertible on $L^p(\partial D)$, $2 \leq p < \infty$, and on $L_1^p(\partial D)$, $1 < p \leq 2$. The single layer potential is shown to be invertible from $L^p(\partial D)$ to $L_1^p(\partial D)$, $1 < p \leq 2$, except for certain domains discussed in Section 4.10 below. The operator $\frac{1}{2}I - K^*$ is shown to be invertible on $L_0^p(\partial D)$, $1 < p \leq 2$. This yields a solution to the Neumann problem when $1 < p \leq 2$. However, it should be noted that Fabes and Kenig had previously observed a proof of the Neumann problem based on conformal mapping. See [16].

To begin we establish the L^2 results. Because of the logarithmic singularity in the single potential in \mathbf{R}^2 the statement of Theorem 2.1 must be modified slightly.

LEMMA 4.1. *For all $f \in L_0^2(\partial D)$*

$$\|(\tfrac{1}{2}I - K^*)f\|_2 \leq C \left\{ \|(\tfrac{1}{2}I + K^*)f\|_2 + \left| \int_{\partial D} Sf \right| \right\}$$

and

$$\|(\tfrac{1}{2}I + K^*)f\|_2 \leq C \left\{ \|(\tfrac{1}{2}I - K^*)f\|_2 + \left| \int_{\partial D} Sf \right| \right\},$$

where C depends only on D .

Proof. If f has mean value zero then

$$Sf(X) = \frac{1}{2\pi} \int_{\partial D} [\log |X - Q| - \log |X|] f(Q) dQ,$$

which decays like $|X|^{-1}$ as $|X| \rightarrow \infty$. Likewise $\nabla Sf(X)$ decays like $|X|^{-2}$. Thus the use of the Gauss divergence theorem may be justified over the unbounded region $\mathbf{R}^2 \setminus \bar{D}$ and Lemma 2.2 may be stated for $n = 2$. Now the proof is the same as that of Section 2.

THEOREM 4.2. *Let D be a bounded Lipschitz domain in \mathbf{R}^2 . Then*

$$(i) \quad \frac{1}{2}I + K: L^2(\partial D) \rightarrow L^2(\partial D) \text{ and}$$

$$(ii) \quad \frac{1}{2}I - K^*: L_0^2(\partial D) \rightarrow L_0^2(\partial D)$$

are invertible operators.

Proof. The proof is virtually that of Section 3.

Note that if $(\frac{1}{2}I + K^*)f = 0$, then $\int_{\partial D} f = \int_{\partial D} (\frac{1}{2}I - K^*)f = 0$ so that $f \in L_0^2(\partial D)$. This fact allows us to apply the Gauss divergence theorem to $\int_{\mathbf{R}^2 \setminus D} |\nabla Sf(X)|^2 dX$ and obtain $Sf = 0$ a.e. on ∂D as in Section 3. Continuing the familiar arguments yields that $\frac{1}{2}I + K^*$ is one to one on $L^2(\partial D)$.

To show that $\frac{1}{2}I + K^*$ has closed range on $L^2(\partial D)$ it suffices to show closed range on $L_0^2(\partial D)$ since the codimension is finite. The familiar arguments of Section 3 apply.

It also suffices to do the dense range argument on $L_0^2(\partial D)$ since

$$\begin{aligned} \int_{\partial D} (\frac{1}{2}I + K^*)(1) &= \int_{\partial D} 1 - \int_{\partial D} (\frac{1}{2}I - K^*)(1) \\ &= \int_{\partial D} 1 \neq 0. \end{aligned}$$

Thus part (i) follows.

Again it is easy to devise arguments showing that $\frac{1}{2}I - K^*$ is one to one with closed range in $L_0^2(\partial D)$.

To show that $\frac{1}{2}I - K^*$ has dense range in $L_0^2(\partial D)$ we repeat the arguments of Theorem 3.3.

4.3. DEFINITION. Recall from the proof of Theorem 3.3 the unique function, f_0 , in $L^2(\partial D)$ such that $(\frac{1}{2}I - K^*)f_0 = 0$ and $\int_{\partial D} f_0 d\sigma = 1$. Define the closed subspace of codimension one

$$\mathcal{W} = \left\{ f \in L^2(\partial D): \int_{\partial D} f f_0 d\sigma = 0 \right\}.$$

Define $\mathcal{W}^p = \mathcal{W} \cap L^p(\partial D)$ for $p \geq 2$.

We state without proof the following corollary to the last theorem.

COROLLARY 4.4. (i) $\frac{1}{2}I + K^*: L^2(\partial D) \rightarrow L^2(\partial D)$,

(ii) $\frac{1}{2}I - K: L^2(\partial D)/\langle 1 \rangle \rightarrow L^2(\partial D)/\langle 1 \rangle$,

(iii) $\frac{1}{2}I + K: \mathcal{W} \rightarrow \mathcal{W}$,

(iv) $\frac{1}{2}I - K: \mathcal{W} \rightarrow \mathcal{W}$

are invertible operators.

Dahlberg's Theorem 0.8D will be used here and in Section 5 to relate the boundary values of a harmonic function with those of its harmonic conjugates. In two dimensions the harmonic conjugate to a double layer potential may itself be written as a potential.

4.5. DEFINITION. Let $N(Q) = (n^1(Q), n^2(Q))$ denote the outer unit normal to D at $Q \in \partial D$. Define the tangent vector $T(Q) = (-n^2(Q), n^1(Q))$ and for any $f \in L^2(\partial D)$

$$\tilde{\mathcal{H}}f(X) = -\frac{1}{2\pi} \int_{\partial D} \frac{\langle Q - X, T(Q) \rangle}{|Q - X|^2} f(Q) dQ.$$

It is easy to see that $\mathcal{H}f$ and $\tilde{\mathcal{H}}f$ satisfy the Cauchy–Riemann equations,

$$\begin{aligned} \frac{\partial}{\partial x_1} \mathcal{H}f &= \frac{\partial}{\partial x_2} \tilde{\mathcal{H}}f, \\ \frac{\partial}{\partial x_2} \mathcal{H}f &= -\frac{\partial}{\partial x_1} \tilde{\mathcal{H}}f, \end{aligned}$$

for all $X = (x_1, x_2) \in \mathbf{R}^2 \setminus \partial D$. In particular $|\nabla \mathcal{H}f(X)| = |\nabla \tilde{\mathcal{H}}f(X)|$.

Recall by Lemma 1.5 that given a family of regular cones $\{\Gamma\}$

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma(P) = \Gamma_i(P) \cup \Gamma_e(P)}} \tilde{\mathcal{H}}f(X) = \left(S \frac{\partial}{\partial T} \right) f(P) \quad \text{a.e.}$$

4.6. DEFINITION. For $f \in \mathcal{W}$ (Section 4.3) define interior and exterior Hilbert transforms by

$$H_i f = \left(S \frac{\partial}{\partial T} \right) \left(\frac{1}{2} I + K \right)^{-1} f$$

and

$$H_e f = - \left(S \frac{\partial}{\partial T} \right) \left(\frac{1}{2} I - K \right)^{-1} f.$$

The operators H_i and H_e are bounded on $L^2(\partial D)$ by Lemma 1.5 and Theorem 4.2. Both map \mathcal{W} into \mathcal{W} by Corollary 4.4 and the fact that $Sf_0(X) = \text{constant}$, $X \in D$, so that for $g \in L^2$

$$\begin{aligned} & \int_{\partial D} f_0(P) \text{p.v.} \int_{\partial D} \frac{\langle Q - P, T(Q) \rangle}{|Q - P|^2} g(Q) dQ dP \\ &= \int_{\partial D} g(Q) \text{p.v.} \int_{\partial D} \frac{\langle Q - P, T(Q) \rangle}{|Q - P|^2} f_0(P) dP dQ \\ &= \int_{\partial D} g(Q) \lim_{\substack{X \rightarrow Q \\ X \in \Gamma_i(Q)}} \langle T(Q), \nabla Sf_0(X) \rangle dQ = 0. \end{aligned}$$

Because

$$H_i f(P) = \lim_{\substack{X \rightarrow P \\ X \in \Gamma_i(P)}} \mathcal{H}(\tfrac{1}{2}I + K)^{-1} f(X)$$

is the boundary value of the harmonic function conjugate to the Poisson extension of f in D , $H_i^2 = -I$ on \mathcal{W} and similarly $H_e^2 = -I$ on \mathcal{W} . This is because the nonzero constant functions are not in \mathcal{W} .

LEMMA 4.7. (i) $H_i: \mathcal{W}^p \rightarrow \mathcal{W}^p$ is bounded and $H_i^2 = -I$ on \mathcal{W}^p , $2 \leq p < \infty$,

(ii) $H_e: \mathcal{W}^p \rightarrow \mathcal{W}^p$ is bounded and $H_e^2 = -I$ on \mathcal{W}^p , $2 \leq p < \infty$.

Proof. Put $u = \mathcal{H}(\tfrac{1}{2}I - K)^{-1}f$ and $\tilde{u} = \mathcal{H}(\tfrac{1}{2}I - K)^{-1}f$. Let $2 \leq p < \infty$ and take $f \in \mathcal{W}^p$. Consider a (large) ball, B , containing \bar{D} . Fix a point $P^* \in B \setminus \bar{D}$. Then

$$|u(P^*)| \leq C \|(\tfrac{1}{2}I - K)^{-1}f\|_2 \leq C \|f\|_2 \leq C \|f\|_p,$$

where the C depends on P^* . Similarly $|\tilde{u}(P^*)| \leq C \|f\|_p$. Likewise for any $Q \in \partial B$, $|u(Q)|$ and $|\tilde{u}(Q)|$ are bounded by a constant times $\|f\|_p$. Now given a regular family of cones, $\{\Gamma\}$, we may apply Theorem 0.8D to $B \setminus \bar{D}$. Recall the definition from Section 0.7 of $A(u, \Gamma)(P) = A(u)(P)$ and that $|\nabla u(X)| = |\nabla \tilde{u}(X)|$.

$$\begin{aligned} \|H_e f\|_{L^p(\partial D)} &\leq C \{ \|(\tilde{u} - \tilde{u}(P^*))^*\|_{L^p(\partial D)} + \|f\|_{L^p(\partial D)} \} \\ &\leq C \{ \|A(\tilde{u})\|_{L^p(\partial D \cup \partial B)} + \|f\|_{L^p(\partial D)} \} \\ &\equiv C \{ \|A(u)\|_{L^p(\partial D \cup \partial B)} + \|f\|_{L^p(\partial D)} \} \\ &\leq C \{ \|u - u(P^*)^*\|_{L^p(\partial D \cup \partial B)} + \|f\|_{L^p(\partial D)} \} \\ &\leq C \{ \|u^*\|_{L^p(\partial D)} + \|f\|_{L^p(\partial D)} \} \leq C \|f\|_{L^p(\partial D)}. \end{aligned}$$

Note that it is in the last inequality that by Theorem 0.9D, $p \geq 2$ is required. Similarly we may work in D and obtain

$$\|H_1 f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)}.$$

LEMMA 4.8. For all $f \in \mathcal{W}^p$, $2 \leq p < \infty$,

$$(i) \quad \|(\tfrac{1}{2}I - K)f\|_{L^p(\partial D)} \leq C \|(\tfrac{1}{2}I + K)f\|_{L^p(\partial D)} \text{ and}$$

$$(ii) \quad \|(\tfrac{1}{2}I + K)f\|_{L^p(\partial D)} \leq C \|(\tfrac{1}{2}I - K)f\|_{L^p(\partial D)},$$

where C depends only on D and p .

Proof.

$$\begin{aligned} \|(\tfrac{1}{2}I - K)f\|_{L^p(\partial D)} &= \|H_e^2(\tfrac{1}{2}I - K)f\|_{L^p(\partial D)} \\ &\leq C \left\| S \frac{\partial}{\partial T} f \right\|_{L^p(\partial D)} = C \|H_1(\tfrac{1}{2}I + K)f\|_{L^p(\partial D)} \\ &\leq C \|(\tfrac{1}{2}I + K)f\|_{L^p(\partial D)}. \end{aligned}$$

Similarly we may show (ii).

THEOREM 4.9. (i) $\tfrac{1}{2}I + K: L^p(\partial D) \rightarrow L^p(\partial D)$, $2 \leq p < \infty$, is invertible,
(ii) $\tfrac{1}{2}I - K^*: L_0^{p'}(\partial D) \rightarrow L_0^{p'}(\partial D)$, $1 < p' \leq 2$, is invertible.

Proof. By the L^2 results $\tfrac{1}{2}I + K$ is one to one and part (i) of Lemma 4.8 together with the usual arguments shows that $\tfrac{1}{2}I + K$ has closed range.

To show dense range let $1/p' + 1/p = 1$ and take $f \in L_0^{p'}(\partial D)$, $(\tfrac{1}{2}I + K^*)f = 0$. Since $Sf \in L^\infty(\mathbf{R}^2)$ and by the Sobolov lemma is continuous on ∂D , the uniqueness result, Lemma 3.7, and the fact that D is regular for the Dirichlet problem assure us that Sf is continuous on all of \mathbf{R}^2 .

Let $\Omega_j \downarrow D$. Since f has mean value zero we may apply the Gauss divergence theorem, obtaining

$$\lim_{j \rightarrow \infty} \int_{\mathbf{R}^2 \setminus \Omega_j} |\nabla Sf(X)|^2 dX = - \lim_{j \rightarrow \infty} \int_{\partial \Omega_j} \frac{\partial Sf}{\partial N_j} Sf d\sigma_j = 0.$$

Thus because Sf decays at ∞ to zero it is in fact zero on all of \mathbf{R}^2 , whence $f = 0$. Recall $f_0 \in L^2(\partial D)$ such that $\int_{\partial D} f_0 d\sigma = 1$ and $(\tfrac{1}{2}I - K^*)f_0 = 0$ (see Definition 4.3). It follows that $f_0 \in L^{p'}(\partial D)$ and $L_0^{p'}(\partial D) \oplus \langle f_0 \rangle = L^{p'}(\partial D)$. Because $(\tfrac{1}{2}I + K^*)f_0 = f_0$ and $\tfrac{1}{2}I + K^*: L_0^{p'} \rightarrow L_0^{p'}$ we have $\tfrac{1}{2}I + K^*$ one to one on all of $L^{p'}(\partial D)$. This completes the proof of part (i).

For part (ii) we consider the space \mathcal{W}^p to be dual to $L_0^{p'}(\partial D)$. Part (ii) of Lemma 4.8 shows by the usual arguments that $\tfrac{1}{2}I - K: \mathcal{W}^p \rightarrow \mathcal{W}^p$ is one to one with closed range.

Now consider $f \in L_0^{p'}(\partial D)$, $(\frac{1}{2}I - K^*)f = 0$. Once again we may justify

$$\int_D |\nabla Sf(X)|^2 dX = 0.$$

This time we conclude $Sf = \text{constant}$ in D . Thus

$$0 = \int_{\partial D} f Sf d\sigma = \int_{\partial D} (\tfrac{1}{2}I + K^*) f Sf d\sigma$$

and as before we may conclude $Sf \equiv 0$ in \mathbf{R}^2 and thus $f = 0$.

4.10. *Remark.* For $D \subset \mathbf{R}^2$ put $D_r = \{X: rX \in D\}$, $r > 0$. For $Q \in \partial D_r$ put $f'_0(Q) = f_0(rQ)$, where f_0 is the function of Section 4.3. In higher dimensions we know that in D , $Sf_0 = \text{constant} \neq 0$. However, the following anomaly occurs in two dimensions. For $X \in D$

$$\begin{aligned} Sf_0(X) &= \frac{1}{2\pi} \int_{\partial D} \log |X - Q| f_0(Q) dQ \\ &= \frac{\log r}{2\pi} \int_{\partial D} f_0(Q) dQ + \frac{r}{2\pi} \int_{\partial D_r} \log |r^{-1}X - Q| f'_0(Q) dQ \quad (1) \\ &\equiv \frac{\log r}{2\pi} + rS_r f'_0(r^{-1}X). \end{aligned}$$

Let K_r denote the double layer boundary potential defined with respect to D_r . Then the formula, (1), shows that $\langle f'_0 \rangle$ is the kernel of $\frac{1}{2}I - K_r^*$. Recall that $Sf_0 = \text{constant}$ in D and $S_r f'_0 = \text{constant}$ in D_r . If $Sf_0 = 0$ in D and if $r \neq 1$, then $S_r f'_0 \neq 0$ in D_r . On the other hand if $Sf_0 \neq 0$ in D then choosing r such that $\log r = 2\pi Sf_0$ implies that $S_r f'_0 = 0$ in D_r . Thus in a sense the domains, D , for which $Sf_0 = 0$ in D are rare. This is to be contrasted with the higher-dimensional cases, where Sf_0 is never zero. With this in mind we turn to regularity for the Dirichlet problem in two dimensions.

THEOREM 4.11. For $D \subset \mathbf{R}^2$ such that $Sf_0 \neq 0$ in D

$$S: L^p(\partial D) \rightarrow L_1^p(\partial D), \quad 1 < p \leq 2,$$

is an invertible operator.

Proof. That S is one to one follows from arguments used in the proof of Theorem 4.9.

Consider now any $g \in L_1^p(\partial D)$. Then $\partial g / \partial T \in L^p(\partial D)$ with mean value zero by a partition of unity argument. Put

$$u(X) = -S \left[(\tfrac{1}{2}I - K^*)^{-1} \frac{\partial g}{\partial T} \right] (X)$$

and let \tilde{u} be the harmonic conjugate of u in D . Then for any $P \in \partial D$ and $X \in \Gamma_i(P)$, $\langle T(P), \nabla u(X) \rangle = \langle -N(P), \nabla \tilde{u}(X) \rangle$ since the Cauchy–Riemann equations are rotation invariant in \mathbf{R}^2 . Thus \tilde{u} solves the Neumann problem with data

$$\frac{\partial}{\partial T} S \left(\frac{1}{2} I - K^* \right)^{-1} \frac{\partial}{\partial T} g \in L_0^p(\partial D).$$

By Theorem 4.9 and the uniqueness result Lemma 3.6,

$$\tilde{u}(X) = -S \left[\left(\frac{1}{2} I - K^* \right)^{-1} \frac{\partial}{\partial T} S \left(\frac{1}{2} I - K^* \right)^{-1} \frac{\partial}{\partial T} g \right] (X).$$

But also $T(P) \cdot \nabla \tilde{u}(X) = N(P) \cdot \nabla u(X) \rightarrow \partial g(P)/\partial T$ a.e. as $X \rightarrow P$, $X \in \Gamma_i(P)$. Thus $\tilde{u}|_{\partial D} - g = \text{constant}$. Since \tilde{u} is given as a single layer potential and $Sf_0 \neq 0$ in D , g is in the range of S .

THEOREM 4.12. *For any $D \subset \mathbf{R}^2$*

$$\frac{1}{2}I + K : L_1^p(\partial D) \rightarrow L_1^p(\partial D), \quad 1 < p \leq 2,$$

is an invertible operator.

Proof. The proof is the same as the proof of (iii) of Theorem 3.3 except that in order to justify the formula used there in the exterior of D we must take $g \in S(L_0^p(\partial D))$, the image under S of L_0^p . Then $(\frac{1}{2}I + K^*)^{-1}S^{-1}g$ has mean value zero since

$$\begin{aligned} \int_{\partial D} (\tfrac{1}{2}I + K^*)^{-1}S^{-1}g \, d\sigma &= \int_{\partial D} S^{-1}g \cdot (\tfrac{1}{2}I + K)^{-1}(1) \, d\sigma \\ &= \int_{\partial D} S^{-1}g \, d\sigma. \end{aligned}$$

Note also that $Sf_0 = 0$ is of no consequence since $(\frac{1}{2}I + K)(1) = 1$.

5. REGULARITY FOR THE DIRICHLET PROBLEM

The purpose of this section is to establish the following.

THEOREM 5.1. *Let D be a bounded Lipschitz domain in \mathbf{R}^n , $n \geq 3$, with connected boundary. Given $1 < p \leq 2$, let $g \in L_1^p(\partial D)$. Then there exists a unique function, u , such that*

- (i) $\Delta u = 0$ in D ,

(ii) $u \rightarrow g$ a.e. in nontangential cones,

(iii) $\|(\nabla u)^*\|_{L^p(\partial D)} < \infty$.

Moreover, $S: L^p(\partial D) \rightarrow L^p_1(\partial D)$ is an invertible operator, i.e., u is given by the classical single layer potential of a unique $L^p(\partial D)$ function, and

$$\|(\nabla u)^*\|_{L^p(\partial D)} \leq C \|g\|_{L^p_1(\partial D)},$$

where C depends only on D , p , and the cones defining $(\nabla u)^*$.

In order to show existence of a harmonic function satisfying (ii) and (iii), we establish an a priori bound

$$\|\partial u / \partial N\|_p \leq C_p \|\nabla_t u\|_p, \quad 1 < p \leq 2, \quad (1)$$

for certain harmonic functions, u , given as single layer potentials of L^2 functions. Recall that ∇_t denotes the tangential gradient on ∂D . This is done in the proof of Lemma 5.4 on the boundaries of starlike domains, Ω , that are essentially cylinders except for a small part of one base that is given as the graph of a Lipschitz function. Given $u = Sf$ with respect to Ω and having established (1) both inside and outside Ω , we may invoke Lemma 1.3 to yield

$$\begin{aligned} \|(\nabla u)^*\|_p &\leq C \|f\|_p \leq C (\|(\tfrac{1}{2}I - K^*)f\|_p + \|(\tfrac{1}{2}I + K^*)f\|_p) \\ &< C \|\nabla_t u\|_p. \end{aligned} \quad (2)$$

When the boundary values of u are supported on the Lipschitz portion of $\partial\Omega$, (1) may be established by utilizing the identity

$$\int_{\partial\Omega} \frac{\partial u}{\partial N} v \, d\sigma = \sum_{j=1}^{n-1} \int_{\partial\Omega} \frac{\partial u}{\partial T_j} v_j \, d\sigma,$$

where the T_j are tangent vectors and the v_j are Stein–Weiss harmonic conjugates to the harmonic function, v , here considered to have boundary values in $L^{p'}(\partial\Omega)$, $1/p + 1/p' = 1$. It is in order to use this identity that we work in the cylindrical domains. That the v_j also have boundary values in $L^{p'}$ is established in the proof of Lemma 5.4 in a way that is reminiscent of arguments used in Section 4 above. There, given \tilde{v} the conjugate harmonic function to v a harmonic function defined in a region of the plane, the Lusin area integral could be easily used to relate the boundary values of v and \tilde{v} since $|\nabla v(X)| \equiv |\nabla \tilde{v}(X)|$. Here this last identity is replaced with a lemma (Lemma 5.2) really due to Elias M. Stein. Stein's lemma essentially says that if

$$\int_{\Gamma(P)} \left(\frac{\partial v_j}{\partial X_k}(X) \right)^2 |X - P|^{2-n} \, dX \quad (3)$$

is “good” for $k = n$ then it is “good” for $1 \leq k \leq n - 1$ also. But $\partial v_j / \partial X_n \equiv \partial v / \partial X_j$ and Dahlberg’s Lusin area integral result says that (3) is “good” for $\partial v / \partial X_j$ in place of $\partial v_j / \partial X_k$.

Having obtained (2) for the cylindrical domains, Ω , we prove Lemmas 5.6 and 5.7 in order to extend (2) to the a priori inequality

$$\|(\nabla u)^*\|_{L^p(\partial D)} \leq C_p \|g\|_{L^p_1(\partial D)}, \quad (4)$$

where D is a general Lipschitz domain and $u \rightarrow g \in L^2_1(\partial D)$ nontangentially. I would like to thank Carlos Kenig for pointing out the utility of lemmas of Dahlberg on the Green’s function and on positive harmonic functions that vanish on portions of the boundary in making this extension of (2). See [6].

Once (4) is established for $g \in L^2_1(\partial D)$, the fact that $S: L^2(\partial D) \rightarrow L^2_1(\partial D)$ is onto (Theorem 3.3) allows us to produce an easy approximation argument that completes the proof of Theorem 5.1.

The proof of the following lemma may be found in [20, pp. 213–216].

LEMMA 5.2. *Let $\varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ be a Lipschitz function with norm not exceeding M . Let γ, Γ , and Γ^* be circular truncated half-cones with axes normal to \mathbf{R}^{n-1} and vertices $(0, 0) \in \mathbf{R}^n$ such that*

$$\gamma \subset \bar{\gamma} \setminus \{(0, 0)\} \subset \Gamma \subset \bar{\Gamma} \setminus \{(0, 0)\} \subset \Gamma^*$$

and for all

$$P = (x_0, \varphi(x_0))$$

$(\Gamma^ + P) \setminus \{P\}$ is contained in the region below the graph of φ . Denote the base of $(\Gamma + P)$ by B_P . Let u be a harmonic function in any bounded or unbounded region, Ω_- , below the graph of φ . Then for all $P = (x_0, \varphi(x_0))$ such that $\Gamma^* + P \subset \Omega_-$,*

$$\int_{\gamma+P} |D_{x_j} u(X)|^2 \frac{dX}{|X - P|^{n-2}} \leq C \left\{ \int_{\Gamma+P} |D_{x_n} u(X)|^2 \frac{dX}{|X - P|^{n-2}} + \sup_{X \in B_P} |D_{x_j} u(X)|^2 \right\}, \quad j = 1, \dots, n-1,$$

where C depends only on n, M, γ, Γ , and Γ^ .*

Let us construct some special Lipschitz domains that will be of use to us.

5.3. DEFINITION. Let $\varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ be a Lipschitz function with norm not exceeding $M < \infty$ and with support contained in $B^{n-1}(0; 1)$ where $B^{n-1}(x; r) = \{y \in \mathbf{R}^{n-1} : |x - y| < r\}$.

Define the one-component cone

$$T = \{(x, s): 0 < s < 9M, |x| < s/M\}.$$

Put $\Omega_+ = [(0, -2M) + T] \cap \{(x, s): s > \varphi(x)\}$ and $\Omega_- = [(0, 2M) - T] \cap \{(x, s): s < \varphi(x)\}$. Let $P_+ = (0, 3M)$, $P_- = (0, -3M)$; define the frustums of the half-cones $(0, \mp 2M) \pm T$ by

$$F_+ = \{(x, s) \in \Omega_+ : s > 2M\}$$

and

$$F_- = \{(x, s) \in \Omega_- : s < -2M\}.$$

Finally define

$$\Omega = \{(x, s): \varphi(x) < s < 10M, |x| < 10\}.$$

Recall that $S: L^2(\partial\Omega) \rightarrow L^2_1(\partial\Omega)$ is by Theorem 3.3 an invertible operator where here S is defined with respect to Ω .

THEOREM 5.4. *With φ, M , and Ω as defined in Section 5.3, let $g \in L^2_1(\partial\Omega)$ with support in*

$$\Delta = \{(x, \varphi(x)): |x| < \delta < \frac{1}{3}\}.$$

Let $f \in L^2(\partial\Omega)$ and $u = Sf$ so that u is the Poisson extension of g in Ω . Then for $1 < p \leq 2$

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \|\nabla_t g\|_{L^p(\partial\Omega)},$$

where C depends only on δ, M, p, Ω , and the cones defining $(\nabla u)^$.*

Proof. We may take $\varphi_j \in C^\infty(\mathbf{R}^{n-1})$ such that $\text{supp } \varphi_j \equiv \overline{B^{n-1}(0; 1)}$, $\varphi_j \uparrow \varphi$ uniformly, $\varphi_j(x) < \varphi(x)$ if $x \in B^{n-1}(0; 1)$, and $\nabla \varphi_j \rightarrow \nabla \varphi$ a.e. and in every L^q , $1 \leq q < \infty$, as in the remarks of Section 0.3. Define

$$\Omega_j = \{(x, s): \varphi_j(x) < s < 10M, |x| < 10\} \supset \Omega.$$

Let v be the single layer potential of an $L^2(\partial\Omega)$ function so that v has positive, continuous boundary values, $h \in C(\partial\Omega)$.

Assume now that there are harmonic functions v_1, \dots, v_{n-1} defined in Ω_- so that together with v we have a conjugate harmonic system in Ω_- , i.e.,

$$\begin{aligned} D_{x_l} v_j(x, s) &= D_{x_j} v_l(x, s), & 1 \leq l, j \leq n-1, \\ D_s v_j(x, s) &= D_{x_j} v(x, s), & 1 \leq j \leq n-1, \end{aligned} \quad (1)$$

$$\sum_{j=1}^{n-1} D_{x_j} v_j(x, s) + D_s v(x, s) = 0, \quad \text{for all } (x, s) \in \Omega_-.$$

Then for $Q_j \in \partial\Omega_j \cap \Omega_-$

$$\frac{\partial}{\partial N} v(Q_j) = - \sum_{l=1}^{n-1} \frac{\partial}{\partial T_l} v_l(Q_j) \equiv - \sum_{l=1}^{n-1} \langle T_l(Q_j), \nabla v_l(Q_j) \rangle,$$

where N denotes the outer unit normal to Ω_j and T_l denotes a tangent vector to Ω_j as defined in Section 1.4.

Since u and v decay rapidly enough at infinity familiar convergence arguments yield

$$\begin{aligned} \int_{\partial\Omega_j} \frac{\partial u}{\partial N} v \, d\sigma_j &= \int_{\partial\Omega_j} u \frac{\partial v}{\partial N} \, d\sigma_j \\ &= \sum_{l=1}^{n-1} \int_{|x| < r} u(x, \varphi_j(x)) \frac{\partial}{\partial x_l} v_l(x, \varphi_j(x)) \, dx \\ &\quad + \int_{r < |x| < 1} u(x, \varphi_j(x)) \langle (\nabla \varphi_j(x), -1), \nabla v(x, \varphi_j(x)) \rangle \, dx, \quad (2) \end{aligned}$$

where $\frac{1}{3} < r < 1$ is chosen so that $\text{supp } \varphi \subset B^{n-1}(0; r)$.

Since $\text{supp}(g) \subset \mathcal{A}$, $u(x, \varphi_j(x))$ for $r \leq |x|$ vanishes continuously as $j \rightarrow \infty$. Thus since v is a single layer potential of an L^2 function the last integral converges to zero.

The integrals under the summation equal

$$\begin{aligned} & - \int_{|x| < r} \frac{\partial}{\partial x_l} u(x, \varphi_j(x)) v_l(x, \varphi_j(x)) \, dx \\ & + \int_{\theta \in \partial B^{n-1}(0; r)} N^l(\theta) u(\theta, \varphi_j(\theta)) v_l(\theta, \varphi_j(\theta)) \, d\theta. \end{aligned} \quad (3)$$

If the v_l have interior nontangential maximal functions, v_l^* , with respect to Ω_- that are in $L^{p'}(\partial\Omega_-)$, then there is an r , $\frac{1}{3} < r < 1$, such that $\int_{\partial B^{n-1}(0; r)} (v_l^*(\theta, \varphi(\theta)))^{p'} \, d\theta < \infty$, $l = 1, \dots, n-1$. This follows because $v_l^* \in L^{p'}(\partial\Omega_-)$ implies that

$$\int_0^1 r^{n-2} \, dr \int_{\partial B^{n-1}(0; 1)} (v_l^*(r\theta, \varphi(r\theta)))^{p'} \, d\theta < \infty$$

so that the inner integral must be finite for a.e. $r \in (0, 1)$. Since for every $x \in B^{n-1}(0; 1)$, $|v_l(x, \varphi_j(X))| \leq v_l^*(x, \varphi(X))$, dominated convergence may be applied to the second integral of (3) for each l . Letting $j \rightarrow \infty$, (2) now yields

$$\left| \int_{\partial\Omega} \left(\frac{1}{2} I + K^* \right) f \cdot h \, d\sigma \right| \leq \sum_l \int_{\Delta} \left| \frac{\partial}{\partial T_l} g \right| v_l^* \, d\sigma. \quad (4)$$

Consequently we wish to show the existence of the v_l in Ω_- and establish that

$$\|v_l^*\|_{L^{p'}(\partial\Omega_-)} \leq C \|h\|_{L^p(\partial\Omega)}. \quad (5)$$

Hence, let the base of $(0, 2M) - T$ be denoted by

$$b = \{(x, s) \in \partial\Omega_- : s = -7M\}$$

and consider b to be the ball of radius 9 centered at the origin in \mathbf{R}^{n-1} . Then for all $(x, s) \in \Omega_-$ define

$$v_l(x, s) = \int_{-7M}^s D_{x_l} v(x, \alpha) d\alpha - \frac{1}{\omega_{n-1}} \int_b \frac{x_l - y_l}{|x - y|^{n-1}} D_l v(y, -7M) dy, \quad (6)$$

$l = 1, \dots, n-1$, where the second integral is a derivative of the Newtonian potential of $D_l v(\cdot, 7M)$. Then (1) is established.

Note that in compact sets bounded away from $\partial\Omega$, e.g., in the fustrum, F_- , the functions v_1, \dots, v_{n-1}, v are pointwise uniformly bounded by $C \|h\|_{L^{p'}(\partial\Omega)}$, where C depends on the compact set but is independent of h . To see this first note that by (6) it suffices to demonstrate this for v . Next put

$$\tilde{v}(X) = |X - P_+|^{2-n} v \left(P_+ + \frac{X - P_+}{|X - P_+|^2} \right) \quad \text{for } P_+ + \frac{X - P_+}{|X - P_+|^2} \in \mathbf{R}^n \setminus \bar{\Omega}.$$

The Kelvin transform of v, \tilde{v} , is harmonic in

$$\tilde{D} = \left\{ X: P_+ + \frac{X - P_+}{|X - P_+|^2} \in \mathbf{R}^n \setminus \bar{\Omega} \right\} \cup \{P_+\} \subset B(P_+; 1)$$

because the singularity at P_+ is removable since v is a single layer potential. The Jacobian relating the surface measures over $\partial\Omega$ and $\partial\tilde{D}$ is bounded uniformly away from zero and infinity. Because we are now in a bounded domain Theorem 0.9D may be used in \tilde{D} to obtain the desired bound for v .

Next we wish to apply Theorem 0.8D in order to show (5). For convenient application of this result we make the following observations.

There are half-cones, γ, Γ , and Γ^* , as in the hypotheses of Lemma 5.2, such that when $(x, s) \in \partial\Omega_- \setminus \partial F_-$ the bases of the half-cones $\Gamma + (x, s)$ form a compact subset of F_- . Next $\{\Gamma + (x, s): (x, s) \in \partial\Omega_- \setminus \partial F_-\}$ and $\{\gamma + (x, s): (x, s) \in \partial\Omega_- \setminus \partial F_-\}$ may be extended to regular families of cones $\{\tilde{\Gamma}\}$ and $\{\tilde{\gamma}\}$ for all of Ω_- in such a way that the added subfamily of cones for $\{\tilde{\gamma}\}$ may itself be extended to a regular family, $\{\tilde{\tilde{\gamma}}\}$, of cones for F_- .

Recall now the notation for nontangential maximal functions introduced in Section 0.6 and the Lusin area integral introduced in Section 0.7. For all $P \in \partial\Omega_- \setminus \partial F$, Lemma 5.2 and (1) yield

$$A(v_l, \tilde{\gamma}_l)(P) \leq C \{A(v, \tilde{\Gamma}_l)(P) + \|h\|_{L^{p'}(\partial\Omega)}\},$$

$l = 1, \dots, n - 1$. Hence by Theorems 0.8D and 0.9D

$$\begin{aligned}
 & \|N(v_l - v_l(P_-), \tilde{\gamma}_l)\|_{L^{p'}(\partial\Omega_-)} \\
 & \leq C \|A(v_l, \tilde{\gamma}_l)\|_{L^{p'}(\partial\Omega_-)} \\
 & \leq C \{\|A(v, \tilde{F}_l)\|_{L^{p'}(\partial\Omega_- \setminus \partial F_-)} + \|h\|_{L^{p'}(\partial\Omega)} \\
 & \quad + \|N(v_l - v_l(P_-), \tilde{\gamma}_l)\|_{L^{p'}(\partial F_-)}\} \\
 & \leq C \{\|N(v - v(P_-), \tilde{F}_l)\|_{L^{p'}(\partial\Omega_- \setminus \partial F_-)} + \|h\|_{L^{p'}(\partial\Omega)}\} \\
 & \leq C \|h\|_{L^{p'}(\partial\Omega)}.
 \end{aligned}$$

The inequality $\|N(v, \tilde{F}_l)\|_{L^{p'}(\partial\Omega_- \setminus \partial F_-)} \leq C \|h\|_{L^{p'}(\partial\Omega)}$ may be justified by Harnack (since $v \geq 0$) and the Kelvin transform as above. Thus (5) is established.

Since $L_1^2(\partial\Omega) \cap C(\partial\Omega)$ is dense in $L^{p'}(\partial\Omega)$, (4) and (5) suffice to prove

$$\|(\tfrac{1}{2}I + K^*)f\|_{L^p(\partial\Omega)} \leq C \|\nabla_t g\|_{L^p(\partial\Omega)}. \quad (7)$$

As easier argument using Ω_+ instead of Ω_- yields

$$\|(\tfrac{1}{2}I - K^*)f\|_{L^p(\partial\Omega)} \leq C \|\nabla_t g\|_{L^p(\partial\Omega)}. \quad (7')$$

That is, conjugate harmonics to v are defined in Ω_+ . Then

$$\left| \int_{\partial\Omega} \left(\frac{1}{2}I - K^* \right) f \cdot h \, d\sigma \right| \leq \sum_l \int_{\Delta} \left| \frac{\partial}{\partial T_l} y \right| v_l^* \, d\sigma$$

is established, where the nontangential maximal functions are defined using cones interior to Ω_+ . Then Theorems 0.8D and 0.9D may be applied as above. The argument is easier only because it is not necessary to use the Kelvin transform in order to justify Dahlberg's theorems in unbounded domains.

Since we know by Lemma 1.3 that with respect to any regular family for Ω

$$\|(\nabla S f)^*\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)},$$

the triangle inequality, (7) and (7') prove the lemma.

5.5. For \mathcal{A} as in Lemma 5.4 put

$$v\mathcal{A} = \{(x, \varphi(x)): |x| \leq v\delta\}, \quad v = 2, 3, \dots$$

LEMMA 5.6. With φ, M, Ω , and \mathcal{A} as in Lemma 5.4, let $g \in L_1^2(\partial\Omega)$ be a positive function supported in \mathcal{A} . Let D be a bounded Lipschitz domain such that

$$\partial D \cap \partial\Omega \supset 3\mathcal{A}.$$

Let $u = Sf, f \in L^2(\partial D)$, so that u is the Poisson extension of g in D , when g is defined to be identically zero on $\partial D \setminus A$. Let $1 < p \leq 2$.

(i) If $\Omega \subset D$, then

$$\|(\nabla u)_e^*\|_{L^p(\partial D)} \leq C \|\nabla_t g\|_{L^p(\partial D)}.$$

(ii) If $\Omega \subset \mathbf{R}^n \setminus \bar{D}$, then

$$\|(\nabla u)_i^*\|_{L^p(\partial D)} \leq C \|\nabla_t g\|_{L^p(\partial D)}.$$

Here the constants depend only on p, M, δ, D, Ω , and the cones defining $(\nabla u)^*$ with respect to D .

Proof. To prove (i) let \tilde{u} be a single layer potential with respect to Ω so that $\tilde{u} \rightarrow g$ on A and zero elsewhere on $\partial\Omega$. By using Theorem 0.9D and the uniqueness lemma, Lemma 3.7, for example, it is possible to show that $\tilde{u} \geq 0$ in $\mathbf{R}^n \setminus \Omega$ and likewise that $\tilde{u} - u \geq 0$ in $\mathbf{R}^n \setminus D$. Again by Lemma 3.7, $\tilde{u} - u$ vanishes continuously on $3A$. (Continuous vanishing is due to the Poisson representation. See [6, 11, 12].) Let $\{I\}$ be regular for D and take $P \in 2A$ and $X \in \Gamma_e(P)$. Then

$$|\nabla u(X)| \leq |\nabla(\tilde{u} - u)(X)| + |\nabla \tilde{u}(X)|.$$

Lemma 5.4 guarantees that

$$\|N(\nabla \tilde{u}, \Gamma_e)\|_{L^p(2A)} \leq C \|\nabla_t g\|_{L^p(\partial D)}.$$

Because $\{I\}$ is regular the mean value property for harmonic functions gives

$$|\nabla(\tilde{u} - u)(X)| \leq C \frac{(\tilde{u} - u)(X)}{|X - P|}. \quad (1)$$

Consider a ball, $B(0; R) \equiv B$ such that $B(0; R/2) \supset D$. Let $G(X, Y)$ denote the Green's function for $B \setminus \bar{D}$. We may choose $A \in B \setminus \bar{D}$ such that $E = \{Q \in 3A \setminus \bar{A}; \Gamma_e(Q) \ni A\}$ has surface measure $\geq C_0 > 0$, where we may take C_0 to depend only on D and $\{I\}$. Then using, for example, [14, Theorem 5.25] or [6, Lemma 8], we may assert

$$(\tilde{u} - u)(X) \leq C \frac{(\tilde{u} - u)(A)}{G(A, P^*)} G(X, P^*), \quad (2)$$

where C depends only on A, D , and $\{I\}$, and P^* is fixed outside $B(0; R/2)$. We have

$$(\tilde{u} - u)(A) \leq \tilde{u}(A) = \int_0^1 \langle \nabla \tilde{u}(tQ + (1-t)A), (Q - A) \rangle dt,$$

for $Q \in E$. An integration over E and Lemma 5.4 yield

$$\begin{aligned} \tilde{u}(A) &\leq C \|N(\nabla \tilde{u}, \Gamma_e)\|_{L^p(3\Delta)} \\ &\leq C \|\nabla_t g\|_{L^p(\partial D)}. \end{aligned} \quad (3)$$

Thus (1), (2), and (3) yield for $X \in \Gamma_e(P)$

$$|\nabla(\tilde{u} - u)(X)| \leq C \|\nabla_t g\|_{L^p(\partial D)} \frac{G(X, P^*)}{|X - P|}.$$

Let ω denote harmonic measure on $\partial\Omega$ with respect to the point P^* . Consider cylinders, $Z(P, r)$, with axis perpendicular to \mathbf{R}^{n-1} and $P \in 2\Delta$. Then by [6, Lemma 1] there are constants C, C_1 depending only on $\{I\}$ such that

$$\frac{G(X, P^*)}{|X - P|} \leq C_1 \frac{\omega(Z(P, C|X - P|) \cap \partial\Omega)}{|X - P|^{n-1}}$$

for all $P \in 2\Delta, X \in \Gamma_e(P)$. By Theorem 0.10D, then,

$$\frac{G(X, P^*)}{|X - P|} \leq C_2 \mathcal{M}(k)(P) \in L^p(\partial\Omega),$$

where \mathcal{M} denotes the Hardy–Littlewood maximal function with respect to surface measure, σ , and

$$d\omega = k \, d\sigma.$$

Thus

$$\begin{aligned} &\|N(\nabla(\tilde{u} - u), \Gamma_e)\|_{L^p(2\Delta)} \\ &\leq C \|\nabla_t g\|_{L^p(\partial D)} \|k\|_{L^p(\partial\Omega)} \\ &\leq C' \|\nabla_t g\|_{L^p(\partial D)}. \end{aligned}$$

In order to complete the proof of (i), we observe that u vanishes continuously on $\partial D \setminus \overline{2\Delta}$ and thus essentially repeat the above arguments for u in place of $\tilde{u} - u$. This time the Green's function of $B \setminus \bar{D}$ is used.

Part (ii) is similarly proved.

LEMMA 5.7. *Let D be a bounded Lipschitz domain in \mathbf{R}^n and $g \in L^2_1(\partial D)$. Put $u = Sf, f \in L^2(\partial D)$ so that u is the Poisson extension of g in D . Then for $1 < p \leq 2$*

$$\|(\nabla u)^*\|_{L^p(\partial D)} \leq C \|g\|_{L^p_1(\partial D)},$$

where C depends on p, D , and the cones defining $(\nabla u)^*$.

Proof. Let (Z_j, φ_j) , $j = 1, \dots, N$, be the coordinate pairs associated with a regular family of cones for D as in Section 0.4. Let $M = 100 \max_j \{\|\nabla \varphi_j\|_{L^\infty \mathbb{R}^{n-1}}\}$. Fix $\varepsilon > 0$ so that for any $Q \in \partial D$ there is a coordinate cylinder, $Z(Q, 100\varepsilon)$, with height equal to $100\varepsilon M$ and such that $Z(Q, 100\varepsilon)$ is contained in some Z_j with axis parallel to the axis of Z_j . Thus put $Q = (x_0, \varphi_j(x_0))$ and define $\varphi: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(x) &= \max\{M|x - x_0| + \varphi_j(x_0) - \varepsilon M, \varphi_j(x)\}, & |x - x_0| < 2\varepsilon, \\ &= \varphi_j(x_0) + \varepsilon M, & |x - x_0| \geq 2\varepsilon. \end{aligned}$$

Define

$$\Omega = \{(x, s): \varphi(x) < s < 31\varepsilon M - \varphi(x_0), |x - x_0| < 30\varepsilon\}$$

and

$$\Delta = \{(x, \varphi(x)): |x - x_0| < \varepsilon/4\} \subset \partial D.$$

Then up to change of scale φ, Ω , and Δ are as in Lemmas 5.6 and 5.4.

Let g_+ denote the positive part of g and $g_- = g_+ - g$. By approximating in L_1^p with smooth functions (see [17]) it follows that

$$\|g_\pm\|_{L_1^p(\partial D)} \leq \|g\|_{L_1^p(\partial D)}.$$

We may cover ∂D with a finite number of the Δ 's constructed above and then construct a C^∞ partition of unity with respect to the covering. Thus it suffices to consider ψg_\pm , where $\psi \in C_0^\infty(\Delta)$ is fixed for each Δ . Since

$$\nabla_t(\psi g_\pm) = \psi \nabla_t g_\pm + \nabla_t \psi g_\pm$$

part (i) of Lemma 5.6 yields

$$\|(\nabla u)_e^*\|_{L^p(\partial D)} \leq C \|g\|_{L_1^p(\partial D)}.$$

Next we argue similarly in order to use (ii) of Lemma 5.6 and obtain $\|(\nabla u)_i^*\|_{L^p(\partial D)} \leq C \|g\|_{L_1^p(\partial D)}$.

Proof of Theorem 5.1. Let $g \in L_1^p(\partial D)$ and let $\{\Gamma\}$ be a family of regular cones for D . There exists a sequence $\{g_j\} \subset L_1^2(\partial D)$ such that $g_j \rightarrow g$ in $L_1^p(\partial D)$. We may denote the Poisson extensions of the g_j by $u_j = Sf_j$ for a sequence $\{f_j\}$ of $L^2(\partial D)$ functions. Hence

$$\begin{aligned} \|f_j - f_k\|_{L^p(\partial D)} &\leq \|(\tfrac{1}{2}I - K^*)(f_j - f_k)\|_{L^p(\partial D)} \\ &\quad + \|(\tfrac{1}{2}I + K^*)(f_j - f_k)\|_{L^p(\partial D)} \\ &\leq \|N(\nabla(u_j - u_k), \Gamma)\|_{L^p(\partial D)} \leq C \|g_j - g_k\|_{L_1^p(\partial D)}, \end{aligned} \tag{1}$$

where Lemma 5.7 is used in the last inequality. Thus $\{f_j\}$ is a Cauchy sequence in $L^p(\partial D)$ so $f_j \rightarrow f$ in $L^p(\partial D)$. Letting $u = Sf$, $u \rightarrow g$ a.e. in nontangential cones since

$$\int_{\partial D} |Sf(Q) - g(Q)|^p dQ = 0.$$

Now let $(\nabla u)^*$ denote the nontangential maximal function of $|\nabla u|$ taken over any fixed family of regular cones. By Lemma 1.3 and arguing as in (1)

$$\begin{aligned} \|(\nabla u)^*\|_{L^p(\partial D)} &\leq C \|f\|_{L^p(\partial D)} \\ &= C \lim_j \|f_j\|_{L^p(\partial D)} \leq C \overline{\lim}_j \|N(\nabla u_j, \Gamma)\|_{L^p(\partial D)} \\ &\leq C \lim_j \|g_j\|_{L^p_1(\partial D)} = C \|g\|_{L^p_1(\partial D)}. \end{aligned}$$

To show uniqueness of u , let $\Delta u = 0$ in D , $u \rightarrow 0$ a.e. nontangentially, and $(\nabla u)^* \in L^p(\partial D)$. Arguing as in [9, pp. 183–184] take $\psi_\varepsilon \in C_0^\infty(D)$ so that $0 \leq \psi_\varepsilon \leq 1$, $\psi_\varepsilon \equiv 1$ on $\{X \in D: \text{distance}(X, \partial D) \geq \varepsilon\}$, and $|D^\alpha \psi_\varepsilon| \leq C_\alpha \varepsilon^{-|\alpha|}$ for $\varepsilon > 0$ and any multi-index, α . Let $G(X; Y)$ denote the Green's function of D . Fixing $X \in D$ we have for small ε ,

$$\begin{aligned} u(X) &= (\psi_\varepsilon u)(X) = \int_D G(X; Y) \Delta(\psi_\varepsilon u)(Y) dY \\ &= 2 \int_D G(X; Y) \langle \nabla \psi_\varepsilon(Y), \nabla u(Y) \rangle dY \\ &\quad + \int_D G(X; Y) u(Y) \Delta \psi_\varepsilon(Y) dY. \end{aligned}$$

The last integral may be dominated by a finite sum of integrals of the form

$$\frac{C'}{\varepsilon^2} \int_{|y| < C} dy \int_0^\varepsilon |G(X; y, t + \varphi(y))| |u(y, t + \varphi(y))| dt, \quad (2)$$

where φ is the Lipschitz function associated with a coordinate cylinder, Z , for D , and $y \in \mathbf{R}^{n-1} \cap Z$.

As in the proof of Lemma 5.6 we write for almost all such y since $u \rightarrow 0$ nontangentially

$$u(y, t + \varphi(y)) = \int_0^1 \langle \nabla u(s(y, t + \varphi(y)) + (1-s)(y, \varphi(y))), (0, t) \rangle ds.$$

Thus taking a suitable average

$$|u(y, t + \varphi(y))| \leq t(\nabla u)^*(y, \varphi(y))$$

and (2) is dominated by

$$C \|(\nabla u)^*\|_{L^p(\partial D)} \cdot \max\{|G(X; Y)|: Y \in \text{support of } \Delta\psi_\varepsilon(Y)\}.$$

Since $G(X; \cdot)$ is continuous up to ∂D , letting $\varepsilon \rightarrow 0$ we conclude

$$\int_D G(X; Y) u(Y) \Delta\psi_\varepsilon(Y) dY \rightarrow 0.$$

Similarly $\int_D G(X; Y) \langle \nabla\psi_\varepsilon(Y), \nabla u(Y) \rangle dY \rightarrow 0$. Thus $u \equiv 0$ in D .

To show that the function, $f \in L^p(\partial D)$, such that $Sf = g$ is unique, assume that $Sf \rightarrow 0$ a.e. in nontangential cones. By the uniqueness result proved in the last paragraph $Sf \equiv 0$ in D , whence $(\frac{1}{2}I - K^*)f = 0$. Now let $B \equiv B(0, R) \supset \bar{D}$. Again Sf is unique among the harmonic functions, v , such that $v|_{\partial B} = Sf|_{\partial B}$, $v \rightarrow 0$ a.e. nontangentially on ∂D , and $(\nabla v)^* \in L^p(\partial B \cup \partial D)$. Letting v be the unique harmonic function in $C(\bar{B} \setminus D): v|_{\partial B} = Sf|_{\partial B}$ and $v|_{\partial D} = 0$ we wish to show that $(\nabla v)^* \in L^p$. But this may be accomplished by comparing v to the Green's function for $B \setminus \bar{D}$ in a manner similar to the arguments of the proof of Lemma 5.6. The only difficulty is that v is not necessarily a positive harmonic function. But if we add to v some constant times the Green's function for $B(0, 2R) \setminus \bar{D}$, where the pole for the Green's function is fixed outside $B(0, R)$, then the comparison goes through. Thus Sf must be continuous up to ∂D so that letting $R \rightarrow \infty$ and invoking the maximum principle imply that $Sf \equiv 0$ in $\mathbf{R}^n \setminus D$. Therefore $(\frac{1}{2}I + K^*)f = 0$ and thus $f = 0$. This completes the proof of Theorem 5.1.

5.8. Remark. By Dahlberg's result [6], harmonic measure, ω^X , at $X \in D$ may be written $d\omega^X(Q) = k^X(Q) d\sigma(Q)$, where $k^X \in L^2(\partial D; d\sigma)$. For $Y \in \mathbf{R}^n \setminus \bar{D}$ inspection shows that

$$S(k^X)(Y) = \frac{1}{\omega_n(2-n)} \frac{1}{|X - Y|^{n-2}}$$

since this is the unique harmonic function in X with boundary values

$$\frac{1}{\omega_n(2-n)} \frac{1}{|Q - Y|^{n-2}}, \quad Q \in \partial D.$$

Thus fixing $X \in D$ and letting $Y \rightarrow Q \in \partial D$, one obtains

$$S(k^X)(Q) = \frac{1}{\omega_n(2-n)} \frac{1}{|X - Q|^{n-2}}.$$

Now let μ be a positive finite, Borel measure on ∂D . Then $S\mu(X)$ is harmonic in D and by Hunt and Wheeden [12] and Dahlberg [6] has nontangential limits $S\mu(Q)$ for a.e. $(d\sigma) Q \in \partial D$.

For convenience assume D starlike with respect to the origin and write $k d\sigma = d\omega = d\omega^0$.

Fatou's lemma and Fubini yield

$$\int_{\partial D} S\mu \, d\omega \leq \lim_{t \uparrow 1} t^{2-n} \int_{\partial D} S k(t^{-1}P) \, d\mu(P) = S\mu(0);$$

i.e., $S\mu \in L^1(\partial D; d\omega)$.

In particular, given any $f \in L_1^p(\partial D)$, $1 < p \leq 2$, $f \in L^1(\partial D; d\omega)$ since f is given as the single layer potential at some L^p function, g . Since L_1^2 is dense in L_1^p we even have a Poisson representation $Sg(X) = \int_{\partial D} f d\omega^X$ whenever $f \in L_1^p$. (Again I would like to thank Carlos Kenig for suggesting that the harmonic functions of Theorem 5.1 would have such representations.) Further, $(Sg)^* \in L^p(\partial D, d\omega)$. To see this recall that by Hunt and Wheeden $(Sg)^*$ is dominated pointwise by the Hardy-Littlewood maximal function with respect to ω . Since $p > 1$ therefore

$$\int_{\partial D} (Sg)^{*p} \, d\omega \leq C \int_{\partial D} |Sg|^p \, d\omega.$$

But $|Sg(Q)|^p \leq C S(|g|^p)(Q)$ and the conclusion follows.

5.9 Remark. If the statement of Theorem 5.1 is made for a given $p > 2$ one can find Lipschitz domains so that the statement is false. One merely has to consider domains for which $k^X \notin L^p$ [7]. The harmonic function

$$G(X, \cdot) - \frac{1}{\omega_n(2-n)} \frac{1}{|X - \cdot|^{n-2}}$$

will then serve as a counterexample.

On the other hand given a Lipschitz domain the theorem actually holds for a small interval of p 's above 2. This is because for a harmonic function, v , $\|v^*\|_{L^p(\partial D)} \leq C \|v\|_{L^p(\partial D)}$ holds for a small interval of p 's below 2 so that Lemma 5.4 still holds.

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